Aldo Maceri

Numerical Sequences and Series

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PREFACE

This e-book is the chapter 4 (Numerical Sequences and Series) of the e-book Mathematical Analysis by Aldo Maceri, published by Accademica s.r.l. (https://www.accademica.eu).

This e-book provides a full section of the knowledge that constitutes the Mathematical Analysis. The subject is treated in a thorough and detailed way, but absolutely clear.

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Chapter 4

Numerical Sequences and Series ◊

4.1 Sequences of real numbers

4.1.1 *Regular sequences*

Definition 4.1.1 A *sequence in* $\mathbb{R}$ (or *numerical sequence*) is a function x that maps the set of the positive integers $\mathbb{N}$ into the set of the real numbers $\mathbb{R}$. For any sequence x, we will write x_n instead of $x(n)$ for the value of x at n. The *real number* x_n is called the n^{th} *term* of the sequence. The sequence x whose n^{th} term is x_n will be denoted by

$$x_1, \ldots, x_n, \ldots$$

or simply $\{x_n\}$. ◊

Theorem 4.1.1 *The unique point of accumulation for* $\mathbb{N}$ *is the symbol* $+\infty$.

Proof. From theorem 3.1.9 it follows that no real number is point of accumulation of $\mathbb{N}$. From theorem 3.2.1 it follows that the symbol $+\infty$ is point of accumulation for $\mathbb{N}$. Finally, from theorem 3.2.2 it follows that the symbol $-\infty$ is not point of accumulation for $\mathbb{N}$. ◊

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Definition 4.1.2 Let $\{x_n\}$ be a sequence of real numbers, $l \in \mathbb{R} \cup \{-\infty, +\infty\}$. We say that $\{x_n\}$ is a *regular sequence*, or that $\{x_n\}$ *has limit* l *when* n *positively diverges*, or that $\{x_n\}$ *has limit* l *when* n *is tending to* $+\infty$, and we write

(4.1.1) $$\lim_{n\to+\infty} x_n = l$$

if

(4.1.2) $$\forall\, I_l \quad \exists\, J_{+\infty} \;:\quad (n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_l)$$

i.e., if for every neighborhood I_l of l there exists a neighborhood $J_{+\infty}$ of the symbol $+\infty$ such that if the index n belongs to $J_{+\infty}$, then the sequence's term x_n belongs to neighborhood I_l.

If $\{x_n\}$ is a regular sequence and $l \in \mathbb{R}$, we say that $\{x_n\}$ is *convergent*, or that $\{x_n\}$ *converges to* l.

If $\{x_n\}$ is a regular sequence and $l = +\infty$, we say that $\{x_n\}$ is *positively divergent*, or that $\{x_n\}$ *positively diverges.*

If $\{x_n\}$ is a regular sequence and $l = -\infty$, we say that $\{x_n\}$ is *negatively divergent*, or that $\{x_n\}$ *negatively diverges.* ◊

Definition 4.1.3 We call *infinitesimal* a sequence of real numbers $\{x_n\}$ that converges to zero. ◊

Theorem 4.1.2 *Let* $\{x_n\}$ *be any sequence of real numbers, convergent to real number* l. *The following statements are equivalent*

1) $\forall I_l \quad \exists J_{+\infty} : \quad (n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_l)$

2) $\forall \varepsilon > 0 \quad \exists \nu \in \mathbb{N} : \forall n \in \mathbb{N}$

$$(n > \nu) \Rightarrow (l - \varepsilon < x_n < l + \varepsilon).$$

Proof. 1) $\Rightarrow$ 2). Suppose true the 1). Let ε be any positive real number. Hence, as observed in remark 3.2.3, $I_l =]l - \varepsilon , l + \varepsilon[$ is a neighborhood of l. Hence, by hypothesis 1), there exists a neighborhood $J_{+\infty}$ of the symbol $+\infty$ such that $(n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_l)$. Hence, by definition 3.2.2, there exists a real number a such that $J_{+\infty} =]a , +\infty[$. Hence $(n \in \mathbb{N} \cap]a , +\infty[) \Rightarrow (x_n \in I_l)$. Hence $\forall n \in \mathbb{N}$ $(n > a) \Rightarrow (x_n \in]l - \varepsilon , l + \varepsilon[)$. By theorem 1.2.4 (*Archimedean property of* $\mathbb{R}$), there exists $\nu \in \mathbb{N}$ such that $\nu > a$. Thus, there exists $\nu \in \mathbb{N}$ such that, $\forall n \in \mathbb{N}$, $n > \nu$ implies $x_n \in]l - \varepsilon , l + \varepsilon[$, hence $l - \varepsilon < x_n < l + \varepsilon$. Thus, the 2) is true.
2) $\Rightarrow$ 1). Suppose true the 2). Let I_l be any neighborhood of l. Hence, by definition 3.1.8, I_l is an open subset of $\mathbb{R}$ such that $l \in I_l$. Hence, by definitions 3.1.5 and 3.1.7, there exists $\varepsilon \in]0, +\infty[$ such that $]l - \varepsilon , l + \varepsilon[\subseteq I_l$. Hence, by hypothesis, $\exists \nu \in \mathbb{N} : \forall n \in \mathbb{N} \quad (n > \nu) \Rightarrow (l - \varepsilon < x_n < l + \varepsilon)$. Putting $J_{+\infty} =]\nu , +\infty[$, by definition 3.2.2 we obtain that $J_{+\infty}$ is a neighborhood of the symbol $+\infty$. Let $n \in \mathbb{N} \cap J_{+\infty}$. Hence $n \in \mathbb{N}$ and $n > \nu$. Hence, by hypothesis, $l - \varepsilon < x_n < l + \varepsilon$, hence $x_n \in]l - \varepsilon , l + \varepsilon[\subseteq I_l$. Thus, the 1) is true. ◊

Remark 4.1.1 By the (1.2.8), in statement 2) of theorem 4.1.2 the inequality

$$l - \varepsilon < x_n < l + \varepsilon$$

is equivalent to

$$|x_n - l| < \varepsilon \ .$$

In fact, if $l - \varepsilon < x_n < l + \varepsilon$ then $-\varepsilon < x_n - l < \varepsilon$, hence, by the (1.2.8), $|x_n - l| < \varepsilon$. If $|x_n - l| < \varepsilon$ then, by the (1.2.8), $-\varepsilon < x_n - l < \varepsilon$; hence $l - \varepsilon < x_n < l + \varepsilon$. ◊

Theorem 4.1.3 *Let* $\{x_n\}$ *be any sequence of real numbers, positively divergent. The following statements are equivalent*

1) $\forall\, I_{+\infty} \quad \exists\, J_{+\infty} \ : \quad (n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_{+\infty})$

2) $\forall\, k > 0 \quad \exists\, \nu \in \mathbb{N} \ : \ \forall\, n \in \mathbb{N} \quad (n > \nu) \Rightarrow (x_n > k)$.

Proof. 1) ⇒ 2). Suppose true the 1). Let $k > 0$. Then, the set $I_{+\infty} =]k\,, +\infty[$ is a neighborhood of the symbol $+\infty$. Hence, by hypothesis, there exists a neighborhood $J_{+\infty}$ of the symbol $+\infty$ such that $(n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_{+\infty})$. Hence, by definition 3.2.2, there exists a real number a such that $J_{+\infty} =]a\,, +\infty[$. By theorem 1.2.4 (*Archimedean property of* $\mathbb{R}$), there exists $\nu \in \mathbb{N}$ such that $\nu > a$. Thus, there exists $\nu \in \mathbb{N}$ such that, $\forall\, n \in \mathbb{N}$, $n > \nu$ implies $x_n \in I_{+\infty} =]k\,, +\infty[$. Thus, the 2) is true.

2) ⇒ 1). Suppose true the 2). Let $I_{+\infty}$ be any neighborhood of the symbol $+\infty$. Hence, by definition 3.2.2, there exists a real number a such that $I_{+\infty} =]a\,, +\infty[$. Obviously there exists a positive real number k such that $k > a$, hence

$]k,+\infty[\ \subseteq I_{+\infty}$. Since $k>0$, by hypothesis $\exists\, \nu \in \mathbb{N}$: $\forall\, n \in \mathbb{N}$ $(n>\nu) \Rightarrow (x_n>k)$. Putting $J_{+\infty} =]\nu\,,+\infty[$, by definition 3.2.2 we obtain that $J_{+\infty}$ is a neighborhood of the symbol $+\infty$. Let $n \in \mathbb{N} \cap J_{+\infty}$, hence $n \in \mathbb{N}$ and $n > \nu$, hence $x_n > k$, hence $x_n \in]k\,,+\infty[\ \subseteq I_{+\infty}$. Thus, the 1) is true. ⋄

Theorem 4.1.4 *Let* $\{x_n\}$ *be any sequence of real numbers, negatively divergent. The following statements are equivalent*

1) $\forall\, I_{-\infty} \quad \exists\, J_{+\infty}$: $(n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_{-\infty})$

2) $\forall\, k>0 \quad \exists\, \nu \in \mathbb{N}$: $\forall\, n \in \mathbb{N} \quad (n>\nu)$

$$\Rightarrow (x_n < -k)\ .$$

Proof. The proof is perfectly analogous to that of theorem 4.1.3. ⋄

Remark 4.1.2 We expose the following examples of real sequences:

- the constant sequence $\{1\}$ trivially satisfies the condition

  $$\forall\, \varepsilon > 0 \quad \exists\, \nu \in \mathbb{N} \ :\ \forall\, n \in \mathbb{N} \quad (n>\nu) \Rightarrow (|1-1|<\varepsilon)$$

  and then, by theorem 4.1.2, converges to its constant value 1
- the sequence $\left\{\frac{1}{n}\right\}$ satisfies the condition

  $$\forall\, \varepsilon > 0 \quad \exists\, \nu \in \mathbb{N} \ :\ \forall\, n \in \mathbb{N} \quad (n>\nu) \Rightarrow \left(\left|\frac{1}{n}-0\right|<\varepsilon\right).$$

  In fact, for every $\varepsilon > 0$, because the *Archimedean property of*

$\mathbb{R}$, there exists $\nu > \frac{1}{\varepsilon}$; hence, if $n > \nu$, it results $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{\nu} < \varepsilon$; hence, by theorem 4.1.2, the sequence converges to zero

- the sequence $\{n\}$ is positively divergent. In fact, for every $k > 0$, because the *Archimedean property of* $\mathbb{R}$, it exists $\nu > k$; hence, if $n > \nu$, it results $n > \nu > k$; hence, by theorem 4.1.3, the sequence $\{n\}$ is positively divergent;
- the oscillating sequence $0,1,0,1,0,1, \ldots$ obviously is not convergent nor divergent. ◊

Theorem 4.1.5 [*uniqueness of the limit*] *Any regular sequence of real numbers has unique limit.*

Proof. Let $\{x_n\}$ be a regular sequence of real numbers and

(4.1.3) $$\lim_{n\to+\infty} x_n = l .$$

Hence $l \in \mathbb{R}$, or $l = +\infty$ or $l = -\infty$.
If $l \in \mathbb{R}$, by absurd we suppose that there exists $l' \in \mathbb{R} - \{l\}$ such that

(4.1.4) $$\lim_{n\to+\infty} x_n = l' .$$

If $l' > l$, we put $\varepsilon = \frac{l'-l}{2}$, hence

(4.1.5) $$l + \varepsilon = l' - \varepsilon .$$

By theorem 4.1.2, the (4.1.3) implies that

(4.1.6) $\exists\, \nu \in \mathbb{N} \;:\; \forall\, n \in \mathbb{N}$

$$(n > \nu) \Rightarrow (l - \varepsilon < x_n < l + \varepsilon)$$

and (4.1.4) implies that

(4.1.7) $\exists\, \nu' \in \mathbb{N} \ : \ \forall\, n \in \mathbb{N}$
$$(n > \nu') \Rightarrow (l' - \varepsilon < x_n < l' + \varepsilon) \, .$$

Thus, calling m any positive integer greater than $\max\{\nu, \nu'\}$, from (4.1.6), (4.1.7), (4.1.5) it follows that the real number x_m is simultaneously greater and less than the real number $l + \varepsilon$. Absurd. Obviously, if $l' < l$, we can achieve an absurd result with an identical reasoning.
Still in the hypothesis $l \in \mathbb{R}$, by absurd we suppose simultaneously possible the (4.1.3) and the condition

(4.1.8) $$\lim_{n \to +\infty} x_n = +\infty \ .$$

Let ε be any real positive number and $k = |l| + \varepsilon$. By theorem 4.1.2, the (4.1.3) implies that

(4.1.9) $\exists\, \nu \in \mathbb{N} \ : \ \forall\, n \in \mathbb{N} \quad (n > \nu)$
$$\Rightarrow (l - \varepsilon < x_n < l + \varepsilon \leq |l| + \varepsilon).$$

By theorem 4.1.3, the (4.1.8) implies that

(4.1.10) $\exists\, \nu' \in \mathbb{N} \ : \ \forall\, n \in \mathbb{N} \quad (n > \nu')$
$$\Rightarrow (x_n > k = |l| + \varepsilon) \ .$$

Thus, calling m any positive integer greater than $\max\{\nu, \nu'\}$, from (4.1.9), (4.1.10) it follows that the real number x_m is simultaneously greater and less than the real number $|l| + \varepsilon$. Absurd. In the others possible cases, the uniqueness of l can be easily similarly proven. ⋄

Definition 4.1.4 We say that a sequence of real numbers $\{x_n\}$ is *bounded above* if there exists $k \in \mathbb{R}$ such that

$$\forall\, n \in \mathbb{N} \qquad x_n \le k\,. \ \diamond$$

Definition 4.1.5 We say that a sequence of real numbers $\{x_n\}$ is *bounded below* if there exists $h \in \mathbb{R}$ such that

$$\forall\, n \in \mathbb{N} \qquad h \le x_n\,. \ \diamond$$

Definition 4.1.6 We say that a sequence of real numbers $\{x_n\}$ is *bounded* if there exist $h\,, k \in \mathbb{R}$ such that

(4.1.11) $$\forall\, n \in \mathbb{N} \qquad h \le x_n \le k\,. \ \diamond$$

Theorem 4.1.6 *Let $\{x_n\}$ be any convergent sequence of real numbers. Then $\{x_n\}$ is bounded.*

Proof. By hypothesis, $\lim_{n\to+\infty} x_n = l \in \mathbb{R}$. Let ε be a positive real number. Then, by theorem 4.1.2, there exists a positive integer ν such that

$$\forall\, n \in \mathbb{N} \quad (n > \nu) \Rightarrow (l - \varepsilon < x_n < l + \varepsilon)\,.$$

Hence, putting $h = \min\{x_1\,, \dots, x_\nu\,, l - \varepsilon\}$ and $k = \max\{x_1\,, \dots, x_\nu\,, l + \varepsilon\}$, we have

$$\forall\, n \in \mathbb{N} \qquad h \le x_n \le k\,. \ \diamond$$

Theorem 4.1.7 [*sign permanence*] *Let* $\{x_n\}$ *be any regular sequence of real numbers. If* $l > 0$ *, there exists a positive integer* ν *such that*

$$(4.1.12) \qquad \forall\, n \in \mathbb{N} \qquad (n > \nu) \Rightarrow (x_n > 0)\,.$$

Proof. If $l \in]0, +\infty[$, let $\varepsilon \in]0, l[$. Hence, taking account of theorem 4.1.2, there exists a positive integer ν such that

$$\forall\, n \in \mathbb{N} \quad (n > \nu) \Rightarrow (x_n > l - \varepsilon > 0)\,.$$

If $l = +\infty$, let $k \in]0, +\infty[$. By theorem 4.1.3, there exists a positive integer ν such that

$$\forall\, n \in \mathbb{N} \qquad (n > \nu) \Rightarrow (x_n > k > 0)\ .\ \diamond$$

Theorem 4.1.8 *Let* $\{x_n\}$ *and* $\{z_n\}$ *be two any sequences of real numbers. If*

$$(4.1.13) \qquad \forall\, n \in \mathbb{N} \qquad x_n \leq z_n$$

and $\{x_n\}$ *positively diverges, then* $\{z_n\}$ *positively diverges.*

Proof. By theorem 4.1.3 we have $\forall\, k > 0 \quad \exists\, \nu \in \mathbb{N} \ :\ \forall\, n \in \mathbb{N} \quad (n > \nu) \Rightarrow (x_n > k)$. Hence $\forall\, k > 0 \quad \exists\, \nu \in \mathbb{N} \ :\ \forall\, n \in \mathbb{N} \quad (n > \nu) \Rightarrow (z_n \geq x_n > k)$. Hence $\forall\, k > 0 \quad \exists\, \nu \in \mathbb{N} \ :\ \forall\, n \in \mathbb{N} \quad (n > \nu) \Rightarrow (z_n > k)$. Hence, by theorem 4.1.3

$$\lim_{n \to +\infty} z_n = +\infty\ .\ \diamond$$