

Aldo Maceri

*Numerical
Sequences and Series*



Accademica

Prof. Ing. Aldo Maceri
Past full Professor of Scienza delle Costruzioni
University of Roma "Roma Tre"
Department of Engineering
Italy

<https://www.aldo-maceri.com>

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PREFACE

This e-book is the chapter 4 (Numerical Sequences and Series) of the e-book Mathematical Analysis by Aldo Maceri, published by Accademica s.r.l. (<https://www.accademica.eu>).

This e-book provides a full section of the knowledge that constitutes the Mathematical Analysis. The subject is treated in a thorough and detailed way, but absolutely clear.

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Aldo Maceri

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4.1 Sequences of real numbers

4.1.1 Regular sequences

Definition 4.1.1 A sequence in \mathbb{R} (or numerical sequence) is a function x that maps the set of the positive integers \mathbb{N} into the set of the real numbers \mathbb{R} . For any sequence x , we will write x_n instead of $x(n)$ for the value of x at n . The real number x_n is called the n^{th} term of the sequence. The sequence x whose n^{th} term is x_n will be denoted by

$$x_1, \dots, x_n, \dots$$

or simply $\{x_n\}$. \diamond

Theorem 4.1.1 The unique point of accumulation for \mathbb{N} is the symbol $+\infty$.

Proof. From theorem 3.1.9 it follows that no real number is point of accumulation of \mathbb{N} . From theorem 3.2.1 it follows that the symbol $+\infty$ is point of accumulation for \mathbb{N} . Finally, from theorem 3.2.2 it follows that the symbol $-\infty$ is not point of accumulation for \mathbb{N} . \diamond

Definition 4.1.2 Let $\{x_n\}$ be a sequence of real numbers, $l \in \mathbb{R} \cup \{-\infty, +\infty\}$. We say that $\{x_n\}$ is a *regular sequence*, or that $\{x_n\}$ *has limit l when n positively diverges*, or that $\{x_n\}$ *has limit l when n is tending to $+\infty$* , and we write

$$(4.1.1) \quad \lim_{n \rightarrow +\infty} x_n = l$$

if

$$(4.1.2) \quad \forall I_l \quad \exists J_{+\infty} : (n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_l)$$

i.e., if for every neighborhood I_l of l there exists a neighborhood $J_{+\infty}$ of the symbol $+\infty$ such that if the index n belongs to $J_{+\infty}$, then the sequence's term x_n belongs to neighborhood I_l .

If $\{x_n\}$ is a regular sequence and $l \in \mathbb{R}$, we say that $\{x_n\}$ is *convergent*, or that $\{x_n\}$ *converges to l* .

If $\{x_n\}$ is a regular sequence and $l = +\infty$, we say that $\{x_n\}$ is *positively divergent*, or that $\{x_n\}$ *positively diverges*.

If $\{x_n\}$ is a regular sequence and $l = -\infty$, we say that $\{x_n\}$ is *negatively divergent*, or that $\{x_n\}$ *negatively diverges*. \diamond

Definition 4.1.3 We call *infinitesimal* a sequence of real numbers $\{x_n\}$ that converges to zero. \diamond

Theorem 4.1.2 Let $\{x_n\}$ be any sequence of real numbers, convergent to real number l . The following statements are equivalent

$$1) \quad \forall I_l \quad \exists J_{+\infty} : (n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_l)$$

$$2) \quad \forall \varepsilon > 0 \quad \exists \nu \in \mathbb{N} : \forall n \in \mathbb{N} \\ (n > \nu) \Rightarrow (l - \varepsilon < x_n < l + \varepsilon).$$

Proof. 1) \Rightarrow 2). Suppose true the 1). Let ε be any positive real number. Hence, as observed in remark 3.2.3, $I_l =]l - \varepsilon, l + \varepsilon[$ is a neighborhood of l . Hence, by hypothesis 1), there exists a neighborhood $J_{+\infty}$ of the symbol $+\infty$ such that $(n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_l)$. Hence, by definition 3.2.2, there exists a real number a such that $J_{+\infty} =]a, +\infty[$. Hence $(n \in \mathbb{N} \cap]a, +\infty]) \Rightarrow (x_n \in I_l)$. Hence $\forall n \in \mathbb{N} (n > a) \Rightarrow (x_n \in]l - \varepsilon, l + \varepsilon])$. By theorem 1.2.4 (*Archimedean property of \mathbb{R}*), there exists $\nu \in \mathbb{N}$ such that $\nu > a$. Thus, there exists $\nu \in \mathbb{N}$ such that, $\forall n \in \mathbb{N}, n > \nu$ implies $x_n \in]l - \varepsilon, l + \varepsilon[$, hence $l - \varepsilon < x_n < l + \varepsilon$. Thus, the 2) is true.

2) \Rightarrow 1). Suppose true the 2). Let I_l be any neighborhood of l . Hence, by definition 3.1.8, I_l is an open subset of \mathbb{R} such that $l \in I_l$. Hence, by definitions 3.1.5 and 3.1.7, there exists $\varepsilon \in]0, +\infty[$ such that $]l - \varepsilon, l + \varepsilon[\subseteq I_l$. Hence, by hypothesis, $\exists \nu \in \mathbb{N} : \forall n \in \mathbb{N} (n > \nu) \Rightarrow (l - \varepsilon < x_n < l + \varepsilon)$. Putting $J_{+\infty} =]\nu, +\infty[$, by definition 3.2.2 we obtain that $J_{+\infty}$ is a neighborhood of the symbol $+\infty$. Let $n \in \mathbb{N} \cap J_{+\infty}$. Hence $n \in \mathbb{N}$ and $n > \nu$. Hence, by hypothesis, $l - \varepsilon < x_n < l + \varepsilon$, hence $x_n \in]l - \varepsilon, l + \varepsilon[\subseteq I_l$. Thus, the 1) is true. \diamond

Remark 4.1.1 By the (1.2.8), in statement 2) of theorem 4.1.2 the inequality

$$l - \varepsilon < x_n < l + \varepsilon$$

is equivalent to

$$|x_n - l| < \varepsilon .$$

In fact, if $l - \varepsilon < x_n < l + \varepsilon$ then $-\varepsilon < x_n - l < \varepsilon$, hence, by the (1.2.8), $|x_n - l| < \varepsilon$. If $|x_n - l| < \varepsilon$ then, by the (1.2.8), $-\varepsilon < x_n - l < \varepsilon$; hence $l - \varepsilon < x_n < l + \varepsilon$. \diamond

Theorem 4.1.3 *Let $\{x_n\}$ be any sequence of real numbers, positively divergent. The following statements are equivalent*

- 1) $\forall I_{+\infty} \quad \exists J_{+\infty} : (n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_{+\infty})$
- 2) $\forall k > 0 \quad \exists \nu \in \mathbb{N} : \forall n \in \mathbb{N} \quad (n > \nu) \Rightarrow (x_n > k) .$

Proof. 1) \Rightarrow 2). Suppose true the 1). Let $k > 0$. Then, the set $I_{+\infty} =]k, +\infty[$ is a neighborhood of the symbol $+\infty$. Hence, by hypothesis, there exists a neighborhood $J_{+\infty}$ of the symbol $+\infty$ such that $(n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_{+\infty})$. Hence, by definition 3.2.2, there exists a real number a such that $J_{+\infty} =]a, +\infty[$. By theorem 1.2.4 (*Archimedean property of \mathbb{R}*), there exists $\nu \in \mathbb{N}$ such that $\nu > a$. Thus, there exists $\nu \in \mathbb{N}$ such that, $\forall n \in \mathbb{N}$, $n > \nu$ implies $x_n \in I_{+\infty} =]k, +\infty[$. Thus, the 2) is true.

2) \Rightarrow 1). Suppose true the 2). Let $I_{+\infty}$ be any neighborhood of the symbol $+\infty$. Hence, by definition 3.2.2, there exists a real number a such that $I_{+\infty} =]a, +\infty[$. Obviously there exists a positive real number k such that $k > a$, hence

$]k, +\infty[\subseteq I_{+\infty}$. Since $k > 0$, by hypothesis $\exists v \in \mathbb{N} : \forall n \in \mathbb{N} (n > v) \Rightarrow (x_n > k)$. Putting $J_{+\infty} =]v, +\infty[$, by definition 3.2.2 we obtain that $J_{+\infty}$ is a neighborhood of the symbol $+\infty$. Let $n \in \mathbb{N} \cap J_{+\infty}$, hence $n \in \mathbb{N}$ and $n > v$, hence $x_n > k$, hence $x_n \in]k, +\infty[\subseteq I_{+\infty}$. Thus, the 1) is true. \diamond

Theorem 4.1.4 *Let $\{x_n\}$ be any sequence of real numbers, negatively divergent. The following statements are equivalent*

- 1) $\forall I_{-\infty} \quad \exists J_{+\infty} : (n \in \mathbb{N} \cap J_{+\infty}) \Rightarrow (x_n \in I_{-\infty})$
- 2) $\forall k > 0 \quad \exists v \in \mathbb{N} : \forall n \in \mathbb{N} (n > v) \Rightarrow (x_n < -k)$.

Proof. The proof is perfectly analogous to that of theorem 4.1.3. \diamond

Remark 4.1.2 We expose the following examples of real sequences:

- the constant sequence $\{1\}$ trivially satisfies the condition $\forall \varepsilon > 0 \quad \exists v \in \mathbb{N} : \forall n \in \mathbb{N} (n > v) \Rightarrow (|1 - 1| < \varepsilon)$ and then, by theorem 4.1.2, converges to its constant value 1
- the sequence $\left\{\frac{1}{n}\right\}$ satisfies the condition

$$\forall \varepsilon > 0 \quad \exists v \in \mathbb{N} : \forall n \in \mathbb{N} (n > v) \Rightarrow \left(\left|\frac{1}{n} - 0\right| < \varepsilon\right).$$

In fact, for every $\varepsilon > 0$, because the *Archimedean property* of

\mathbb{R} , there exists $\nu > \frac{1}{\varepsilon}$; hence, if $n > \nu$, it results $\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \frac{1}{\nu} < \varepsilon$; hence, by theorem 4.1.2, the sequence converges to zero

- the sequence $\{n\}$ is positively divergent. In fact, for every $k > 0$, because the *Archimedean property* of \mathbb{R} , it exists $\nu > k$; hence, if $n > \nu$, it results $n > \nu > k$; hence, by theorem 4.1.3, the sequence $\{n\}$ is positively divergent;
- the oscillating sequence $0, 1, 0, 1, 0, 1, \dots$ obviously is not convergent nor divergent. \diamond

Theorem 4.1.5 [*uniqueness of the limit*] *Any regular sequence of real numbers has unique limit.*

Proof. Let $\{x_n\}$ be a regular sequence of real numbers and

$$(4.1.3) \quad \lim_{n \rightarrow +\infty} x_n = l.$$

Hence $l \in \mathbb{R}$, or $l = +\infty$ or $l = -\infty$.

If $l \in \mathbb{R}$, by absurd we suppose that there exists $l' \in \mathbb{R} - \{l\}$ such that

$$(4.1.4) \quad \lim_{n \rightarrow +\infty} x_n = l'.$$

If $l' > l$, we put $\varepsilon = \frac{l' - l}{2}$, hence

$$(4.1.5) \quad l + \varepsilon = l' - \varepsilon.$$

By theorem 4.1.2, the (4.1.3) implies that

$$(4.1.6) \quad \exists \nu \in \mathbb{N} : \forall n \in \mathbb{N}$$

$$(n > \nu) \Rightarrow (l - \varepsilon < x_n < l + \varepsilon)$$

and (4.1.4) implies that

$$(4.1.7) \quad \exists \nu' \in \mathbb{N} : \forall n \in \mathbb{N} \\ (n > \nu') \Rightarrow (l' - \varepsilon < x_n < l' + \varepsilon).$$

Thus, calling m any positive integer greater than $\max\{\nu, \nu'\}$, from (4.1.6), (4.1.7), (4.1.5) it follows that the real number x_m is simultaneously greater and less than the real number $l + \varepsilon$. Absurd. Obviously, if $l' < l$, we can achieve an absurd result with an identical reasoning.

Still in the hypothesis $l \in \mathbb{R}$, by absurd we suppose simultaneously possible the (4.1.3) and the condition

$$(4.1.8) \quad \lim_{n \rightarrow +\infty} x_n = +\infty.$$

Let ε be any real positive number and $k = |l| + \varepsilon$. By theorem 4.1.2, the (4.1.3) implies that

$$(4.1.9) \quad \exists \nu \in \mathbb{N} : \forall n \in \mathbb{N} \quad (n > \nu) \\ \Rightarrow (l - \varepsilon < x_n < l + \varepsilon \leq |l| + \varepsilon).$$

By theorem 4.1.3, the (4.1.8) implies that

$$(4.1.10) \quad \exists \nu' \in \mathbb{N} : \forall n \in \mathbb{N} \quad (n > \nu') \\ \Rightarrow (x_n > k = |l| + \varepsilon).$$

Thus, calling m any positive integer greater than $\max\{\nu, \nu'\}$, from (4.1.9), (4.1.10) it follows that the real number x_m is simultaneously greater and less than the real number $|l| + \varepsilon$. Absurd. In the others possible cases, the uniqueness of l can be

easily similarly proven. \diamond

Definition 4.1.4 We say that a sequence of real numbers $\{x_n\}$ is *bounded above* if there exists $k \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N} \quad x_n \leq k . \diamond$$

Definition 4.1.5 We say that a sequence of real numbers $\{x_n\}$ is *bounded below* if there exists $h \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N} \quad h \leq x_n . \diamond$$

Definition 4.1.6 We say that a sequence of real numbers $\{x_n\}$ is *bounded* if there exist $h, k \in \mathbb{R}$ such that

$$(4.1.11) \quad \forall n \in \mathbb{N} \quad h \leq x_n \leq k . \diamond$$

Theorem 4.1.6 Let $\{x_n\}$ be any convergent sequence of real numbers. Then $\{x_n\}$ is bounded.

Proof. By hypothesis, $\lim_{n \rightarrow +\infty} x_n = l \in \mathbb{R}$. Let ε be a positive real number. Then, by theorem 4.1.2, there exists a positive integer ν such that

$$\forall n \in \mathbb{N} \quad (n > \nu) \Rightarrow (l - \varepsilon < x_n < l + \varepsilon) .$$

Hence, putting $h = \min\{x_1, \dots, x_\nu, l - \varepsilon\}$ and $k = \max\{x_1, \dots, x_\nu, l + \varepsilon\}$, we have

$$\forall n \in \mathbb{N} \quad h \leq x_n \leq k . \diamond$$

Theorem 4.1.7 [sign permanence] *Let $\{x_n\}$ be any regular sequence of real numbers. If $l > 0$, there exists a positive integer ν such that*

$$(4.1.12) \quad \forall n \in \mathbb{N} \quad (n > \nu) \Rightarrow (x_n > 0).$$

Proof. If $l \in]0, +\infty[$, let $\varepsilon \in]0, l[$. Hence, taking account of theorem 4.1.2, there exists a positive integer ν such that

$$\forall n \in \mathbb{N} \quad (n > \nu) \Rightarrow (x_n > l - \varepsilon > 0).$$

If $l = +\infty$, let $k \in]0, +\infty[$. By theorem 4.1.3, there exists a positive integer ν such that

$$\forall n \in \mathbb{N} \quad (n > \nu) \Rightarrow (x_n > k > 0) . \diamond$$

Theorem 4.1.8 *Let $\{x_n\}$ and $\{z_n\}$ be two any sequences of real numbers. If*

$$(4.1.13) \quad \forall n \in \mathbb{N} \quad x_n \leq z_n$$

and $\{x_n\}$ positively diverges, then $\{z_n\}$ positively diverges.

Proof. By theorem 4.1.3 we have $\forall k > 0 \quad \exists \nu \in \mathbb{N} : \forall n \in \mathbb{N} \quad (n > \nu) \Rightarrow (x_n > k)$. Hence $\forall k > 0 \quad \exists \nu \in \mathbb{N} : \forall n \in \mathbb{N} \quad (n > \nu) \Rightarrow (z_n \geq x_n > k)$. Hence $\forall k > 0 \quad \exists \nu \in \mathbb{N} : \forall n \in \mathbb{N} \quad (n > \nu) \Rightarrow (z_n > k)$. Hence, by theorem 4.1.3

$$\lim_{n \rightarrow +\infty} z_n = +\infty . \diamond$$