CHAPTER 8

SEQUENCES AND SERIES OF FUNCTIONS [◊]

8.1 Sequences of functions

8.1.1 Uniform convergence

Definition 8.1.1 Let

- $X \subseteq \mathbb{R}$
- $\forall n \in \mathbb{N} \quad f_n : X \to \mathbb{R}$.

We say that the sequence of functions $\{f_n\}$ *converges on* X (or *converges pointwise on* X) if $\forall x \in X$ the numerical sequence $\{f_n(x)\}$ converges, *i.e.*

(8.1.1) $$\forall x \in X \qquad \lim_{n\to+\infty} f_n(x) \in \mathbb{R}\,.$$ ◊

Definition 8.1.2 Let

- $X \subseteq \mathbb{R}$
- $\forall n \in \mathbb{N} \quad f_n : X \to \mathbb{R}$
- $\{f_n\}$ be convergent on X
- $f : x \in X \to f(x) = \lim_{n\to+\infty} f_n(x) \in \mathbb{R}\,.$

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We call f the *limit* (or the *limit function*) of $\{f_n\}$. ◊

Remark 8.1.1 Let

- $X \subseteq \mathbb{R}$
- $\forall n \in \mathbb{N} \quad f_n : X \to \mathbb{R}$
- $\{f_n\}$ be convergent on X .

Obviously, the statements

(8.1.2) $$f : x \in X \to f(x) = \lim_{n \to +\infty} f_n(x) \in \mathbb{R}$$

(8.1.3) $$\forall x \in X \text{ it results: } \forall \varepsilon > 0 \quad \exists\, \nu \in \mathbb{N} : \quad \forall n \in \mathbb{N} \quad (n > \nu) \Rightarrow (|f_n(x) - f(x)| < \varepsilon)$$

are equivalent. ◊

Definition 8.1.3 Let

- $X \subseteq \mathbb{R}$
- $\forall n \in \mathbb{N} \quad f_n : X \to \mathbb{R}$.

We say that $\{f_n\}$ *converges uniformly on* X if

(8.1.4) $$\forall \varepsilon > 0 \quad \exists\, \nu \in \mathbb{N} : \quad \forall n \in \mathbb{N} \text{ and } \forall x \in X \quad (n > \nu) \Rightarrow (|f_n(x) - f(x)| < \varepsilon) .$$ ◊

Remark 8.1.2 Obviously, any sequence of functions uniformly convergent is convergent. Simple examples prove that the vice versa is false. ◊

Theorem 8.1.1 *Let*

- $X \subseteq \mathbb{R}$
- $\forall n \in \mathbb{N} \quad f_n : X \to \mathbb{R}$
- $\{f_n\}$ *be uniformly convergent on* X
- $x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$ *be accumulation point of* X
- $\forall n \in \mathbb{N} \qquad \lim_{x \to x_0} f_n(x) \in \mathbb{R}$.

Then

(8.1.5) *the numerical sequence* $\left\{ \lim_{x \to x_0} f_n(x) \right\}$ *converges*

(8.1.6) $$f = \lim_{x \to x_0} f_n \text{ converges at } x_0$$

(8.1.7) $$\lim_{n \to +\infty} \lim_{x \to x_0} f_n(x) = \lim_{x \to x_0} \lim_{n \to +\infty} f_n(x) .$$

Proof. We consider the case $x_0 \in \mathbb{R}$, but precise that if $x_0 \in \{-\infty, +\infty\}$ we can reason in similar way.
To prove the (8.1.5), we use the *Cauchy*'s convergence criterion for numerical sequences (*i.e.* theorem 4.1.30). So, putting

(8.1.8) $$\forall n \in \mathbb{N} \qquad l_n = \lim_{x \to x_0} f_n(x) ,$$

it is enough to prove that

(8.1.9) $$\forall \varepsilon > 0 \quad \exists\, \nu \in \mathbb{N} \ : \ \forall n, m \in \mathbb{N} \quad (n > \nu \ \text{ and } \ m > \nu) \Rightarrow (|l_n - l_m| < \varepsilon)\,.$$

Let $\varepsilon > 0$. By the uniform convergence of $\{f_n\}$, there exists $\nu \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ and $\forall x \in X$ it results $(n > \nu) \Rightarrow \left(|f_n(x) - f(x)| < \frac{\varepsilon}{4}\right)$. Hence, $\forall n, m \in \mathbb{N}$ such that $n > \nu$ and $m > \nu$ it results $\forall x \in X$ $|f_n(x) - f_m(x)| \le |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\varepsilon}{2}$. From this, doing tend x to x_0, we immediately obtain $|l_n - l_m| \le \frac{\varepsilon}{2} < \varepsilon$.
Now the (8.1.5) is true and then we can put

(8.1.10) $$\lim_{n \to +\infty} l_n = l \in \mathbb{R}\,.$$

So, to gain the (8.1.6) and (8.1.7) it is enough to prove that

$$\lim_{x \to x_0} f(x) = l\,,$$

i.e. that

(8.1.11) $$\forall \varepsilon > 0 \quad \exists\, \delta > 0 \ : \ \forall x \in X \quad (0 < |x - x_0| < \delta) \Rightarrow (|f(x) - l| < \varepsilon)\,.$$

Preliminary we notice that, by the uniform convergence of $\{f_n\}$

(8.1.12) $$\forall \varepsilon > 0 \quad \exists\, \nu_1 \in \mathbb{N} \ : \ \forall x \in X \ \text{ and } \ \forall n \in \mathbb{N} \quad (n > \nu_1) \Rightarrow \left(|f_n(x) - f(x)| < \frac{\varepsilon}{3}\right)\,;$$

that, by the (8.1.10)

(8.1.13) $\forall \varepsilon > 0 \quad \exists\, \nu_2 \in \mathbb{N} \;:\; \forall n \in \mathbb{N} \quad (n > \nu_2) \Rightarrow \left(|l_n - l| < \frac{\varepsilon}{3}\right)$;

that, by the (8.1.8), putting $\nu = \max\{\nu_1, \nu_2\}$, it results $\lim_{x \to x_0} f_{\nu+1}(x) = l_{\nu+1}$, *i.e.*

(8.1.14) $\forall \varepsilon > 0 \quad \exists\, \delta > 0 \;:\; \forall x \in X$
$$(0 < |x - x_0| < \delta) \Rightarrow \left(|f_{\nu+1}(x) - l_{\nu+1}| < \frac{\varepsilon}{3}\right).$$

Now we can build the (8.1.11). Let $\varepsilon > 0$. The (8.1.14) gives us $\delta > 0$. Let $x \in X$ such that $0 < |x - x_0| < \delta$. We notice that

$$|f(x) - l| \leq |f(x) - f_{\nu+1}(x)| + |f_{\nu+1}(x) - l_{\nu+1}| + |l_{\nu+1} - l|$$

and that

- since $\nu + 1 > \nu \geq \nu_1$, by the (8.1.12) it results
$$|f_{\nu+1}(x) - f(x)| < \frac{\varepsilon}{3}$$
- since the (8.1.14), it results $|f_{\nu+1}(x) - l_{\nu+1}| < \frac{\varepsilon}{3}$
- since $\nu + 1 > \nu \geq \nu_2$, by the (8.1.12) it results $|l_n - l| < \frac{\varepsilon}{3}$.

Hence $|f(x) - l| < \varepsilon$. ◊

Remark 8.1.3 From theorem 8.4.1 it immediately follows that the limit function of a uniformly convergent sequence of continuous functions, is continuous. ◊

Theorem 8.1.2 [*Cauchy* criterion] *Let*

- $X \subseteq \mathbb{R}$
- $\forall n \in \mathbb{N} \quad f_n : X \to \mathbb{R}$.

The following statements are equivalent

(8.1.15) $\{f_n\}$ *is uniformly convergent*

(8.1.16) $$\forall \varepsilon > 0 \quad \exists\, \nu \in \mathbb{N} : \quad \forall n, m \in \mathbb{N} \text{ and } \forall x \in X \quad (n > \nu \text{ and } m > \nu) \Rightarrow (|f_n(x) - f_m(x)| < \varepsilon) .$$

Proof. (8.1.15) ⇒ (8.1.16). We denote f the limit function of $\{f_n\}$. By hypothesis

(8.1.17) $$\forall \varepsilon > 0 \quad \exists\, \nu \in \mathbb{N} : \quad \forall n \in \mathbb{N} \text{ and } \forall x \in X \quad (n > \nu) \Rightarrow \left(|f_n(x) - f(x)| < \frac{\varepsilon}{2}\right) .$$

To build the (8.1.16), we consider any $\varepsilon > 0$. We assume the positive integer ν given by (8.1.17), then choose any $x \in X$ and any $n, m \in \mathbb{N}$ such that $n > \nu$ and $m > \nu$. It results

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \varepsilon .$$

(8.1.16) ⇒ (8.1.15). By the *Cauchy*' convergence criterion (theorem 4.1.30), $\forall x \in X$ the numerical sequence $\{f_n(x)\}$ converges. Putting

$$f = \lim_{n\to+\infty} f_n ,$$

we have to prove that $\{f_n\}$ is uniformly convergent, *i.e.* that

$$\forall\varepsilon > 0 \quad \exists\, \nu \in \mathbb{N} \;:\; \forall n \in \mathbb{N} \text{ and } \forall x \in X$$
$$(n > \nu) \Rightarrow (|f_n(x) - f(x)| < \varepsilon) .$$

So, let $\varepsilon > 0$. Since $(8.1.16)$ $\exists\, \nu \in \mathbb{N}$ such that $\forall x \in X$ and $\forall n \in \mathbb{N}$ with $n > \nu$ it results

(8.1.18) $\qquad \forall m \in \mathbb{N}$ with $m > \nu \qquad |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$.

The $(8.1.18)$, when m tends to $+\infty$, gives us $|f_n(x) - f(x)| \le \frac{\varepsilon}{2} < \varepsilon$ and then the $(8.1.15)$ is true. ◊

Theorem 8.1.3 *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $\forall n \in \mathbb{N} \quad f_n \;:\; [a, b] \to \mathbb{R}$
- $\{f_n\}$ *be uniformly convergent.*

Then, putting $f = \lim_{n\to+\infty} f_n$, *the sequence of functions* $\left\{\int_a^x f_n(t)\, dt\right\}$ *uniformly converges to* $\int_a^x f(t)\, dt$ *on* $[a, b]$, *i.e., uniformly*

(8.1.19) $$\lim_{n\to+\infty} \int_a^x f_n(t)\, dt = \int_a^x \lim_{n\to+\infty} f_n\,(t)\, dt .$$

Proof. We preliminary notice that f is continuous and then there exists

$$g \,:\, x \in [a,b] \;\to\; \int_a^x \lim_{n\to+\infty} f_n(t)\, dt\,.$$

By hypothesis,

(8.1.20) $$\forall \varepsilon > 0 \quad \exists\, \nu \in \mathbb{N} \,:\, \forall n \in \mathbb{N} \text{ and } \forall x \in [a,b] \quad (n > \nu) \Rightarrow (|f_n(x) - f(x)| < \varepsilon)\,.$$

We have to prove that

(8.1.21) $$\forall \varepsilon > 0 \quad \exists\, \nu \in \mathbb{N} \text{ such that, } \forall x \in [a,b] \text{ and } \forall n \in \mathbb{N} \text{ such that } n > \nu \text{, it results } \left|\int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt\right| < \varepsilon\,.$$

To build the (8.1.21), we choose any $\varepsilon > 0$. In correspondence of $\frac{\varepsilon}{2(b-a)}$ the (8.1.20) gives us $\nu \in \mathbb{N}$ such that, for every $t \in [a,b]$ and for every $n \in \mathbb{N}$ such that $n > \nu$, it results

(8.1.22) $$|f_n(t) - f(t)| < \frac{\varepsilon}{2(b-a)}\,.$$

Then, for every $x \in [a,b]$ and for every $n \in \mathbb{N}$ such that $n > \nu$, by (8.1.22) it results

$$\left|\int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt\right| = \left|\int_a^x [f_n(t) - f(t)]\,dt\right|$$

$$\leq \int_a^x |f_n(t) - f(t)|\, dt \leq \int_a^x \frac{\varepsilon}{2(b-a)}\, dt \leq \frac{\varepsilon}{2} < \varepsilon\, . \;\diamond$$

Remark 8.1.4 Known counter-examples show that for uniformly convergent sequences of functions differentiable on $[a, b]$, it may not be true that $\forall x \in [a, b]$

$$\lim_{n\to+\infty} f_n'(x) = \left(\lim_{n\to+\infty} f_n(x)\right)'\, . \;\diamond$$

8.2 Series of functions

8.2.1 Uniform convergence

Definition 8.2.1 Let

- $X \subseteq \mathbb{R}$
- $\forall n \in \mathbb{N} \quad f_n : X \to \mathbb{R}$.

We call *series of functions on X of general term f_n* the sum

(8.2.1) $$\sum_{n=1}^{+\infty} f_n = f_1 + \cdots + f_n + \cdots \quad . \;\diamond$$

Definition 8.2.2 Let

- $X \subseteq \mathbb{R}$
- $\forall n \in \mathbb{N} \quad f_n : X \to \mathbb{R}$.

We say that the series $\sum_{n=1}^{+\infty} f_n$ of functions on X is *convergent* if $\forall x \in X$ the numerical series $\sum_{n=1}^{+\infty} f_n(x)$ is convergent.
If $\sum_{n=1}^{+\infty} f_n$ is a convergent series of functions on X , we call

- *partial sum of the series* the function

$$S_n \ : \ x \in X \ \to \sum_{i=1}^{n} f_i(x)$$

- *sum of the series* the function

$$f \ : \ x \in X \ \to \sum_{n=1}^{+\infty} f_n(x) \quad . \ \diamond$$

Remark 8.2.1 Obviously, if $\sum_{n=1}^{+\infty} f_n$ is a convergent series of functions on X , then

$$\forall x \in X \qquad f(x) = \lim_{n \to +\infty} S_n(x)$$

i.e.

(8.2.2) $\forall x \in X$ and $\forall \varepsilon > 0 \quad \exists\, \nu \in \mathbb{N} \ : \ \forall n \in \mathbb{N}$

$$(n > \nu) \Rightarrow (|S_n(x) - f(x)| < \varepsilon) \ . \ \diamond$$

Definition 8.2.3 Let

- $X \subseteq \mathbb{R}$