CHAPTER 5

LIMITS AND CONTINUITY [◊]

5.1 Limits of real functions of one real variable

5.1.1 Introduction

Definition 5.1.1 Let $\mathbb{R}$ be the set of the real numbers, X be any set contained in $\mathbb{R}$. We call *real function of one real variable* any *single-valued* function f that *maps* the set X *into* $\mathbb{R}$. So, f is a law of correspondence that associates, with each element x of X, a unique real number, which we denote by $f(x)$. The set X is called the *domain* (of *definition*) of f and is denoted by $domf$. We also say that f is *defined* on X. The element $f(x)$ is called the *value* of f in x (or the *image* of *x under f*). The set $f(X)$ of all values of f is called the *range* of f (or *image* of X) and is also denoted $rngf$. If *f maps* $X \subseteq \mathbb{R}$ *into* $\mathbb{R}$ we write

$$f: X \subseteq \mathbb{R} \to \mathbb{R}$$

or

$$f: x \in X \subseteq \mathbb{R} \to f(x) \in \mathbb{R}\,. \quad \diamond$$

Remark 5.1.1 Obviously in general an element of $f(X)$ is the value of *f* at several elements of X . ◊

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Definition 5.1.2 If $f: X \subseteq \mathbb{R} \to Y \subseteq \mathbb{R}$ and $f(X) = Y$, we say that *f* is a function *from X onto Y* . ⋄

Remark 5.1.2 A function is also called *single-valued mapping* or *transformation* or *operation* or *correspondence* or *application.* ⋄

Remark 5.1.3 Let *f* be a function that maps $X \subseteq \mathbb{R}$ into $\mathbb{R}$. Let O, x, y be, in a plane, a system of orthogonal *Cartesian* axes. Consider, for each $x \in X$, the point P of the plane having abscissa x and ordinate $y = f(x)$. As P varies in X, the point $P = (x, f(x))$ describes in the plane a set of points called *Cartesian diagram of the function* f . Orthogonally projecting that diagram on the x-axis we obtain X. Orthogonally projecting that diagram on the y-axis we obtain $f(X)$. ⋄

Definition 5.1.3 Let *f* be a function that maps $X \subseteq \mathbb{R}$ into $\mathbb{R}$. The function

$$\forall x \in X \ \ \to \ |f(x)| \in [0, +\infty[$$

is called the *absolute value* function and is denoted by $|f|$. ⋄

Definition 5.1.4 Let f_1 be a function that maps $X_1 \subseteq \mathbb{R}$ into $\mathbb{R}$, f_2 be a function that maps $X_2 \subseteq \mathbb{R}$ into $\mathbb{R}$. Suppose $X = X_1 \cap X_2 \neq \emptyset$.
The function

$$\forall x \in X \ \ \to \ f_1(x) + f_2(x) \in \mathbb{R}$$

is called the *addition* function and is denoted by $f_1 + f_2$.
The function

$$\forall x \in X \ \ \to \ f_1(x) - f_2(x) \in \mathbb{R}$$

is called the *subtraction* function and is denoted by $f_1 - f_2$.
The function

$$\forall x \in X \ \rightarrow \ f_1(x) \cdot f_2(x) \in \mathbb{R}$$

is called the *multiplication* function and is denoted by $f_1 \cdot f_2$.
The function

$$\forall x \in X - \{x \in X : \ f_2(x) = 0\} \ \rightarrow \ \frac{f_1(x)}{f_2(x)} \in \mathbb{R}$$

is called the *division* function and is denoted by $\frac{f_1}{f_2}$. ◊

Definition 5.1.5 A function *f* that maps $X \subseteq \mathbb{R}$ into $Y \subseteq \mathbb{R}$ is said to be *monotonically increasing* if

(5.1.1) $$\forall y, z \in X \qquad y < z \Rightarrow f(y) \leq f(z) \ .$$ ◊

Definition 5.1.6 A function *f* that maps $X \subseteq \mathbb{R}$ into $Y \subseteq \mathbb{R}$ is said to be *strictly monotonically increasing* if

(5.1.2) $$\forall y, z \in X \qquad y < z \Rightarrow f(y) < f(z) \ .$$ ◊

Definition 5.1.7 A function *f* that maps $X \subseteq \mathbb{R}$ into $Y \subseteq \mathbb{R}$ is said to be *monotonically decreasing* if

(5.1.3) $$\forall y, z \in X \qquad y < z \Rightarrow f(y) \geq f(z) \ .$$ ◊

Definition 5.1.8 A function *f* that maps $X \subseteq \mathbb{R}$ into $Y \subseteq \mathbb{R}$ is said to be *strictly monotonically decreasing* if

(5.1.4) $\forall y, z \in X \qquad y < z \Rightarrow f(y) > f(z)$. ⋄

Definition 5.1.9 Let *f* be a function that maps $X \subseteq \mathbb{R}$ into $\mathbb{R}$. We say that *f* is a *monotone* function if it is monotonically increasing or strictly monotonically increasing or monotonically decreasing or strictly monotonically decreasing. ⋄

Definition 5.1.10 Let *f* be a function that maps $X \subseteq \mathbb{R}$ into $\mathbb{R}$. If

$$\forall x, z \in X \qquad x \neq z \Rightarrow f(x) \neq f(z),$$

we say that *f* is a *reversible* function. ⋄

Remark 5.1.4 Let f be any reversible function that maps $X \subseteq \mathbb{R}$ into $\mathbb{R}$. Obviously $\forall y \in rngf$ there exists one and only one $x \in X$ such that $f(x) = y$. ⋄

Definition 5.1.11 Let *f* be a function that maps $X \subseteq \mathbb{R}$ into $\mathbb{R}$. If f is a reversible function, the (single-valued) function

$$\forall y \in rngf \; \rightarrow \text{ the unique } x \in X \text{ such that } f(x) = y$$

is called the *inverse* function of *f* and is denoted by f^{-1}. ⋄

Remark 5.1.5 If $f: X \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a reversible function, obviously $domf^{-1} = rngf$ and $rngf^{-1} = domf$. ⋄

Definition 5.1.12 Let f be a function that maps $X \subseteq \mathbb{R}$ onto $Y \subseteq \mathbb{R}$. If *f* is a reversible function, we say that *f* is a *one to one* (or *biunique*) *correspondence from* $X \subseteq \mathbb{R}$ *onto* $Y \subseteq \mathbb{R}$. ⋄

Thus, to say that f is a one to one (or biunique) correspondence from X onto Y simply means that each element of Y is the correspondent (by f) of one and only one element of X and each element of X is the correspondent (by f^{-1}) of one and only one element of Y.

Remark 5.1.6 Obviously, if $f: X \subseteq \mathbb{R} \to \mathbb{R}$ is a strictly monotone function, then f is a reversible function. ◊

Definition 5.1.13 Let $f: X \subseteq \mathbb{R} \to Y \subseteq \mathbb{R}$ and $g: Y \subseteq \mathbb{R} \to Z \subseteq \mathbb{R}$ be any functions. We define the *composite* function $g \circ f$ to be the function

$$x \in X \subseteq \mathbb{R} \to g \circ f(x) = g\big(f(x)\big) \in Z \subseteq \mathbb{R}\,. \quad ◊$$

Definition 5.1.14 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. We say that f is an *even* function if

$$(x \in X) \quad \Rightarrow \quad (-x \in X)$$
$$\forall x \in X \qquad f(x) = f(-x)\,. \quad ◊$$

Definition 5.1.15 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. We say that f is an *odd* function if

$$(x \in X) \quad \Rightarrow \quad (-x \in X)$$
$$\forall x \in X \qquad f(x) = -f(-x)\,. \quad ◊$$

Definition 5.1.16 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. If there exists a $\beta \in \mathbb{R}$ such that $f(x) \leq \beta$ for every $x \in X$, we say that f is *bounded above*, and call β an *upper bound* for f. ◊

Definition 5.1.17 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. If there exists an $\alpha \in \mathbb{R}$ such that $\alpha \leq f(x)$ for every $x \in X$, we say that f is *bounded below*, and call α a *lower bound* for f. ◊

Definition 5.1.18 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. If f has both an upper bound and a lower bound, then we say that f is a *bounded* function. ◊

Definition 5.1.19 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. By a *maximum* of f we mean an element of $f(X)$, denoted $\max f$ (or $\max_{x \in X} f(x)$), such that $\max f$ is an upper bound for f. ◊

Remark 5.1.7 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. It is obvious that f can have at most one maximum. ◊

Definition 5.1.20 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. By a *minimum* of f we mean an element of $f(X)$, denoted $\min f$ (or $\min_{x \in X} f(x)$), such that $\min f$ is a lower bound for f. ◊

Remark 5.1.8 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. It is obvious that f can have at most one minimum. ◊

We have proven in chapter 1 that the real number set $\mathbb{R}$ has the *completeness property*, that is:

- Let B be any subset of $\mathbb{R}$, not empty. If B is bounded above, there exists the *least upper bound* of B. In other words, there exists one and only one real number e'', called *supremum* of B and denoted $\sup B$, such that:
    - e'' is an upper bound for B

- $\forall \varepsilon \in]0, +\infty[\ \ \exists b \in B \ : \ b > e'' - \varepsilon$.

➢ Let A be any subset of $\mathbb{R}$, not empty. If A is bounded below, there exists the *greatest lower bound* of A. In other words, there exists one and only one real number e', called *infimum* of A and denoted $\inf A$, such that:
- e' is a lower bound for A
- $\forall \varepsilon \in]0, +\infty[\ \ \exists a \in A \ : \ a < e' + \varepsilon$.

Definition 5.1.21 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$, bounded above. We call *supremum* (or *least upper bound*) of f, and denote

$$\sup f \ \ (\text{or } \sup_{x \in X} f(x))$$

the real number $\sup f(X)$. Obviously, $\sup f$ is such that

(5.1.5) $\sup f$ is an upper bound for f,
(5.1.6) $\forall \varepsilon \in]0, +\infty[$ there exists $\bar{x} \in X$ such that $f(\bar{x}) > (\sup f) - \varepsilon$. ⋄

Remark 5.1.9 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$, bounded above. We underline that the real number $\sup f$ may or may not belong to $f(X)$. ⋄

Remark 5.1.10 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$, bounded above. It is clear that f has one, and only one, supremum. ⋄

Definition 5.1.22 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$, bounded below. We call *infimum* (or *greatest lower bound*) of f, and denote

$$\inf f \ \ (\text{or } \inf_{x \in X} f(x))$$

the real number $\inf f(X)$. Obviously, $\inf f$ is such that

(5.1.7) $\inf f$ is a lower bound for f,
(5.1.8) $\forall \varepsilon \in]0, +\infty[$ there exists $\bar{\bar{x}} \in X$ such that $f(\bar{\bar{x}}) < (\inf f) + \varepsilon$. ◊

Remark 5.1.11 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$, bounded below. We underline that the real number $\inf f$ may or may not belong to $f(X)$. ◊

Remark 5.1.12 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$, bounded below. It is clear that f has one, and only one, infimum. ◊

We have seen in chapter 1 that it is convenient to adjoin to $\mathbb{R}$ two new elements, *which are not real numbers*. These objects, called *infinity* and *minus infinity*, are two symbols. They, respectively, are denoted $+\infty$ (or simply ∞) and $-\infty$. The set

$$[-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$$

was called the *extended real number set*. Moreover, about such symbols, we have defined

(5.1.9) $\forall x \in \mathbb{R} \quad -\infty < x < +\infty$
(5.1.10) $+\infty + \infty = +\infty$
(5.1.11) $(-\infty) + (-\infty) = -\infty - \infty = -\infty$
(5.1.12) $\forall x \in \mathbb{R} \quad x + (+\infty) = (+\infty) + x = x - (-\infty) = +\infty$
(5.1.13) $\forall x \in \mathbb{R} \quad x + (-\infty) = (-\infty) + x = x - (+\infty) = -\infty$
(5.1.14) $(+\infty) \cdot (+\infty) = (-\infty) \cdot (-\infty) = +\infty$
(5.1.15) $(+\infty) \cdot (-\infty) = (-\infty) \cdot (+\infty) = -\infty$
(5.1.16) $\forall x \in]0, +\infty[\quad (+\infty) \cdot x = x \cdot (+\infty) = +\infty$

(5.1.17) $\forall x \in]0, +\infty[\quad (-\infty) \cdot x = x \cdot (-\infty) = -\infty$
(5.1.18) $\forall x \in]-\infty, 0[\quad (+\infty) \cdot x = x \cdot (+\infty) = -\infty$
(5.1.19) $\forall x \in]-\infty, 0[\quad (-\infty) \cdot x = x \cdot (-\infty) = +\infty$
(5.1.20) $\forall x \in \mathbb{R} \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0$.

Remark 5.1.13 We underline that *we do not define* $+\infty - \infty = +\infty + (-\infty), -\infty + \infty, \frac{+\infty}{+\infty}, \frac{-\infty}{+\infty}, \frac{+\infty}{-\infty}, \frac{-\infty}{-\infty}$. ◊

Definition 5.1.23 If $f: X \subseteq \mathbb{R} \to \mathbb{R}$ is not bounded above, then we say that $+\infty$ is the *supremum* of f and write

$$\sup f = +\infty. \quad \diamond$$

Definition 5.1.24 If $f: X \subseteq \mathbb{R} \to \mathbb{R}$ is not bounded below, then we say that $-\infty$ is the *infimum* of f and write

$$\inf f = -\infty. \quad \diamond$$

Let $x_0 \in \mathbb{R}$. With definition 3.1.8, we have seen in chapter 3 that every *neighborhood* of (the real number) x_0 is an *open* set N_{x_0} such that

(5.1.21) $$N_{x_0} \subseteq \mathbb{R} , \quad x_0 \in N_{x_0} .$$

Let $x_0 = +\infty$. By definition 3.2.2, every *neighborhood* of (the symbol) x_0 is an *open* set N_{x_0} such that

(5.1.22) $$N_{x_0} =]a, +\infty[= \{x \in \mathbb{R} : x > a\} , \quad \text{where } a \in \mathbb{R} .$$

Let $x_0 = -\infty$. By definition 3.2.2, every *neighborhood* of (the symbol) x_0 is an *open* set N_{x_0} such that

(5.1.23) $$N_{x_0} =]-\infty, b[\ = \{x \in \mathbb{R} : x < b\}\,, \quad \text{where } b \in \mathbb{R}\ .$$

Let $X \subseteq \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be an *accumulation point* of X. By definition 3.1.12, for *every* neighborhood $N_{x_0}$of x_0 it results

(5.1.24) $$N_{x_0} \cap X - \{x_0\} \neq \emptyset\,.$$

Let $X \subseteq \mathbb{R}$ and let the symbol $x_0 = +\infty$ be an *accumulation point* (to infinity) of X. By definition 3.3.3, for *every* neighborhood $N_{x_0}$of x_0 it results

(5.1.25) $$N_{x_0} \cap X \neq \emptyset\,.$$

Let $X \subseteq \mathbb{R}$ and let the symbol $x_0 = -\infty$ be an *accumulation point* (to infinity) of X. By definition 3.3.3, for *every* neighborhood $N_{x_0}$of x_0 it results

(5.1.26) $$N_{x_0} \cap X \neq \emptyset\,.$$

Let $X \subseteq \mathbb{R}$ and let $x_0 \in \mathbb{R} \cup \{+\infty, -\infty\}$ be an *accumulation point* of X. By (5.1.24), (5.1.25), (5.1.26), for *every* neighborhood $N_{x_0}$of x_0 it results

(5.1.27) $$N_{x_0} \cap X - \{x_0\} \neq \emptyset\,.$$

Remark 5.1.14 Every finite (that is, belonging to $\mathbb{R}$) accumulation point of the set $X \subseteq \mathbb{R}$ can be or can not be a point of X. In fact, reasoning as in remark 3.2.2, we see that 0 and 1 are both accumulation point of the set $[0,1[\ \subseteq \mathbb{R}$. ◊

5.1.2 Limits

Definition 5.1.25 Let f be a real function of domain $X \subseteq \mathbb{R}$, $l \in \mathbb{R} \cup \{+\infty, -\infty\}$. Let $x_0 \in \mathbb{R} \cup \{+\infty, -\infty\}$ be an *accumulation point* of X. We say that f *tends* to l (or *has limit* l) as x *tends* to x_0, and write

(5.1.28) $$\lim_{x \to x_0} f(x) = l$$

if for *every* neighborhood N_l of l there *exists* a neighborhood N_{x_0} of x_0 such that for *every* $x \in N_{x_0} \cap X - \{x_0\}$ it results $f(x) \in N_l$:

(5.1.29) $$\forall N_l \quad \exists N_{x_0} \ : \ \left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) \in N_l)\ .$$ ◊

Definition 5.1.26 Let f be a real function of domain $X \subseteq \mathbb{R}$, $l \in \mathbb{R} \cup \{+\infty, -\infty\}$. Let $x_0 \in \mathbb{R} \cup \{+\infty, -\infty\}$ be an *accumulation point* of X. Suppose

$$\lim_{x \to x_0} f(x) = l\ .$$

If $l \in \mathbb{R}$, we say that f *converges* to l as x *tends* to x_0 . If $l = +\infty$, we say that f *positively diverges* as x *tends* to x_0 . If $l = -\infty$, we say that f *negatively diverges* as x *tends* to x_0 . In each of these three cases, we say that f is *regular* in x_0 . ◊

Remark 5.1.15 If in (5.1.28) $l \in \{-\infty, +\infty\}$, we also say that f has *infinite limit* in x_0 . If in (5.1.28) $x_0 \in \{-\infty, +\infty\}$, we also say that f has limit l *at infinity*. ◊

The limit definition (5.1.29) admits some equivalent formulation, provided by following theorems.

Theorem 5.1.1 *Let* f *be a real function of domain* $X \subseteq \mathbb{R}$, $l \in \mathbb{R}$. *Let* $x_0 \in \mathbb{R}$ *be an accumulation point of* X. *The following statements are equivalent:*

(5.1.30) $$\forall N_l \quad \exists N_{x_0} \ : \ \left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) \in N_l)$$

(5.1.31) $$\forall \varepsilon \in]0, +\infty[\quad \exists \delta \in]0, +\infty[\ : \ \forall x \in X \quad (0 < |x - x_0| < \delta) \Rightarrow (|f(x) - l| < \varepsilon)$$

(5.1.32) $$\forall \varepsilon \in]0, +\infty[\quad \exists \delta \in]0, +\infty[\ : \ \forall x \in X - \{x_0\} \quad (x_0 - \delta < x < x_0 + \delta) \Rightarrow (l - \varepsilon < f(x) < l + \varepsilon).$$

Proof. (5.1.30) $\Rightarrow$ (5.1.31). Suppose the (5.1.30) true. To obtain the (5.1.31), we choose any $\varepsilon \in]0, +\infty[$. Then, $N_l =]l - \varepsilon, l + \varepsilon[$ is a neighborhood of l. By (5.1.30), it exists a neighborhood $N_{x_0}$of x_0 such that

(5.1.33) $$\left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) \in]l - \varepsilon, l + \varepsilon[).$$

Since N_{x_0} is a neighborhood of $x_0 \in \mathbb{R}$, there exist $h, k \in \mathbb{R}$ such that $N_{x_0} =]h, k[$ and $h < x_0 < k$. So, $\delta = min\{k - x_0, x_0 - h\} \in]0, +\infty[$. Choose $x \in X$ such that $0 < |x - x_0| < \delta$. Since $0 < |x - x_0|$, we have $x \neq x_0$. Since $|x - x_0| < \delta$, we have $-\delta < x - x_0 < \delta$, hence $h \leq x_0 - \delta < x < x_0 + \delta \leq k$, hence $x \in]h, k[= N_{x_0}$. So, $x \in N_{x_0} \cap X - \{x_0\}$. By (5.1.30), we have $f(x) \in N_l =]l - \varepsilon, l + \varepsilon[$. Hence $-\varepsilon < f(x) - l < \varepsilon$. Hence $|f(x) - l| < \varepsilon$.
(5.1.31) $\Rightarrow$ (5.1.32). Suppose the (5.1.31) true. To obtain the (5.1.32), we choose any $\varepsilon \in]0, +\infty[$. By (5.1.31) there exists $\delta \in]0, +\infty[$ such that

(5.1.34) $$\forall x \in X \qquad (0 < |x - x_0| < \delta) \Rightarrow (|f(x) - l| < \varepsilon).$$

Choose $x \in X - \{x_0\}$ such that $x_0 - \delta < x < x_0 + \delta$. Hence $-\delta < x - x_0 < \delta$, hence $0 < |x - x_0| < \delta$. By (5.1.31) $|f(x) - l| < \varepsilon$, hence $l - \varepsilon < f(x) < l + \varepsilon$.
(5.1.32) ⇒ (5.1.30). Suppose the (5.1.32) true. To obtain the (5.1.30), we choose any neighborhood N_l of $l \in \mathbb{R}$. Then, there exist $h, k \in \mathbb{R}$ such that $N_l =]h, k[$ and $h < l < k$. So, $\varepsilon = min\{k - l, l - h\} \in]0, +\infty[$. By (5.1.32), there exists $\delta \in]0, +\infty[$ such that $\forall x \in X - \{x_0\}$

(5.1.35) $$(x_0 - \delta < x < x_0 + \delta) \Rightarrow (l - \varepsilon < f(x) < l + \varepsilon) .$$

We denote $N_{x_0}$the neighborhood $]x_0 - \delta, x_0 + \delta[$ of x_0 and consider any $x \in N_{x_0} \cap X - \{x_0\}$. Obviously $x \in X - \{x_0\}$. Moreover, $x \in N_{x_0} =]x_0 - \delta, x_0 + \delta[$. Hence $x_0 - \delta < x < x_0 + \delta$. By (5.1.35), $h \leq l - \varepsilon < f(x) < l + \varepsilon \leq k$. Hence $f(x) \in]h, k[= N_l$. ⋄

Remark 5.1.16 By the (1.2.8), if $l, \varepsilon, y \in \mathbb{R}$ it results

$$(l - \varepsilon < y < l + \varepsilon) \Leftrightarrow (|y - l| < \varepsilon) .$$

In fact, if $l - \varepsilon < y < l + \varepsilon$ then $-\varepsilon < y - l < \varepsilon$, hence, by the (1.2.8), $|y - l| < \varepsilon$. If $|y - l| < \varepsilon$ then, by the (1.2.8), $-\varepsilon < y - l < \varepsilon$; hence $l - \varepsilon < y < l + \varepsilon$. ⋄

Theorem 5.1.2 *Let f be a real function of domain $X \subseteq \mathbb{R}$, $l = +\infty$. Let $x_0 \in \mathbb{R}$ be an accumulation point of X. The following statements are equivalent:*

(5.1.36) $$\forall N_l \quad \exists N_{x_0} \ : \ \left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) \in N_l)$$

(5.1.37) $\forall k \in \,]0,+\infty[\quad \exists \delta \in \,]0,+\infty[\;:\; \forall x \in X$
$$(0 < |x - x_0| < \delta) \Rightarrow (f(x) > k).$$

Proof. $(5.1.36) \Rightarrow (5.1.37)$. Suppose the (5.1.36) true. To obtain the (5.1.37), we choose any $k \in \,]0,+\infty[$. Then, $N_l = \,]k,+\infty[$ is a neighborhood of the symbol $l = +\infty$. By (5.1.36), it exists a neighborhood $N_{x_0}$of x_0 such that

(5.1.38) $$(x \in N_{x_0} \cap X - \{x_0\}) \Rightarrow (f(x) > k).$$

Since N_{x_0} is a neighborhood of $x_0 \in \mathbb{R}$, there exist $a, b \in \mathbb{R}$ such that $N_{x_0} = \,]a,b[$ and $a < x_0 < b$. So, $\delta = min\{b - x_0, x_0 - a\} \in \,]0,+\infty[$. Choose any $x \in X$ such that $0 < |x - x_0| < \delta$. Since $0 < |x - x_0|$, we have $x \neq x_0$. Since $|x - x_0| < \delta$, we have $a \leq x_0 - \delta < x < x_0 + \delta \leq b$, hence $x \in \,]a,b[\, = N_{x_0}$. So, $x \in N_{x_0} \cap X - \{x_0\}$. By (5.1.38), we have $f(x) > k$.
$(5.1.37) \Rightarrow (5.1.36)$. Suppose the (5.1.37) true. To obtain the (5.1.36), we choose any neighborhood N_l of the symbol $l = +\infty$. Then, there exist $k \in \mathbb{R}$ such that $N_l = \,]k,+\infty[$. By (5.1.37), in correspondence of $|k| + 1 \in \,]0,+\infty[$, there exists $\delta \in \,]0,+\infty[$ such that $\forall x \in X$

(5.1.39) $$(0 < |x - x_0| < \delta) \Rightarrow (f(x) > |k| + 1) .$$

We consider the neighborhood $N_{x_0} = \,]x_0 - \delta, x_0 + \delta[$ of x_0. Choose any $x \in N_{x_0} \cap X - \{x_0\}$. Hence $x \in X$, $x \neq x_0$, and $0 < |x - x_0|$. Moreover, $x \in N_{x_0} = \,]x_0 - \delta, x_0 + \delta[$. Hence $|x - x_0| < \delta$. By (5.1.39) it results $f(x) > |k| + 1 > k$. As a consequence, $f(x) \in N_l$. ⬨

Theorem 5.1.3 *Let f be a real function of domain $X \subseteq \mathbb{R}$, $l = -\infty$. Let $x_0 \in \mathbb{R}$ be an accumulation point of X. The following statements* *are equivalent:*

(5.1.40) $\forall N_l \quad \exists N_{x_0} \ : \ \left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) \in N_l)$

(5.1.41) $\forall k \in]0,+\infty[\quad \exists \delta \in]0,+\infty[\ : \ \forall x \in X$
$(0 < |x - x_0| < \delta) \Rightarrow (f(x) < -k).$

Proof. The demonstration is very similar to that of theorem 5.1.2. ⋄

Theorem 5.1.4 *Let f be a real function of domain $X \subseteq \mathbb{R}$, $l \in \mathbb{R}$. Let $x_0 = +\infty$ be an accumulation point of X. The following statements are equivalent:*

(5.1.42) $\forall N_l \quad \exists N_{x_0} \ : \ \left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) \in N_l)$

(5.1.43) $\forall \varepsilon \in]0,+\infty[\quad \exists \delta \in]0,+\infty[\ : \ \forall x \in X$
$(x > \delta) \Rightarrow (|f(x) - l| < \varepsilon)$.

Proof. $(5.1.42) \Rightarrow (5.1.43)$. Suppose the (5.1.42) true. To obtain the (5.1.43), we choose any $\varepsilon \in]0,+\infty[$. Then, $N_l =]l - \varepsilon, l + \varepsilon[$ is a neighborhood of the real number l. By (5.1.42), it exists a neighborhood $N_{x_0}$of x_0 such that

(5.1.44) $\left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) \in]l - \varepsilon, l + \varepsilon[).$

Since N_{x_0} is a neighborhood of the symbol $x_0 = +\infty$, there exist $h \in \mathbb{R}$ such that $N_{x_0} =]h,+\infty[$. So, $\delta = |h| + 1 \in]0,+\infty[$. Choose $x \in X$ such that $x > \delta$. Hence $x > h$, hence $x \in N_{x_0}$. Since $x \notin \{+\infty\}$, it results $x \in N_{x_0} \cap X - \{x_0\}$. By (5.1.44), we have $f(x) \in]l - \varepsilon, l + \varepsilon[$. Hence $-\varepsilon < f(x) - l < \varepsilon$. Hence $|f(x) - l| < \varepsilon$.

(5.1.43) $\Rightarrow$ (5.1.42). Suppose the (5.1.43) true. To obtain the (5.1.42), we choose any neighborhood N_l of $l \in \mathbb{R}$. Then, there exist $h, k \in \mathbb{R}$ such that $N_l =]h, k[$ and $h < l < k$. So, $\varepsilon = min\{k - l, l - h\} \in]0, +\infty[$. By (5.1.43), there exists $\delta \in]0, +\infty[$ such that $\forall x \in X$

(5.1.45) $$(x > \delta) \Rightarrow (|f(x) - l| < \varepsilon) .$$

We denote N_{x_0} the neighborhood $]\delta, +\infty[$ of x_0 and consider any $x \in N_{x_0} \cap X - \{x_0\}$. Obviously $x \in X - \{x_0\}$. Moreover $x \in N_{x_0} =]\delta, +\infty[$, hence $x > \delta$. By (5.1.45), $h \leq l - \varepsilon < f(x) < l + \varepsilon \leq k$. Hence $f(x) \in]h, k[= N_l$. ◊

Theorem 5.1.5 *Let f be a real function of domain $X \subseteq \mathbb{R}$, $l \in \mathbb{R}$. Let $x_0 = -\infty$ be an accumulation point of X. The following statements are equivalent:*

(5.1.46) $$\forall N_l \quad \exists N_{x_0} \ : \ \left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) \in N_l)$$

(5.1.47) $$\forall \varepsilon \in]0, +\infty[\quad \exists \delta \in]0, +\infty[\ : \ \forall x \in X$$
$$(x < -\delta) \Rightarrow (|f(x) - l| < \varepsilon) .$$

Proof. The demonstration is very similar to that of theorem 5.1.4. ◊

Theorem 5.1.6 *Let f be a real function of domain $X \subseteq \mathbb{R}$, $l = +\infty$. Let $x_0 = +\infty$ be an accumulation point of X. The following statements are equivalent:*

(5.1.48) $$\forall N_l \quad \exists N_{x_0} \ : \ \left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) \in N_l)$$

(5.1.49) $$\forall k \in]0, +\infty[\quad \exists \delta \in]0, +\infty[\ : \ \forall x \in X$$

$$(x > \delta) \Rightarrow (f(x) > k)\,.$$

Proof. (5.1.48) ⇒ (5.1.49). Suppose the (5.1.48) true. To obtain the (5.1.49), we choose any $k \in]0,+\infty[$. Then, $N_l =]k,+\infty[$ is a neighborhood of the symbol l. By (5.1.48), it exists a neighborhood $N_{x_0}$of the symbol x_0 such that

(5.1.50) $$\left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) > k).$$

Since N_{x_0} is a neighborhood of the symbol $x_0 = +\infty$, there exist $h \in \mathbb{R}$ such that $N_{x_0} =]h,+\infty[$. So, $\delta = |h| + 1 \in]0,+\infty[$. Choose $x \in X$ such that $x > \delta$. Hence $x > h$, hence $x \in N_{x_0}$. Since $x \notin \{+\infty\}$, it results $x \in N_{x_0} \cap X - \{x_0\}$. By (5.1.50), we have $f(x) > k$.
(5.1.49) ⇒ (5.1.48). Suppose the (5.1.49) true. To obtain the (5.1.48), we choose any neighborhood N_l of $l = +\infty$. Then, there exist $h \in \mathbb{R}$ such that $N_l =]h = +\infty[$. So, $k = |h| + 1 \geq 1 > 0$. Hence $k \in]0,+\infty[$. By (5.1.49), there exists $\delta \in]0,+\infty[$ such that $\forall x \in X$

(5.1.51) $$(x > \delta) \Rightarrow (f(x) > k)\,.$$

We denote $N_{x_0}$the neighborhood $]\delta,+\infty[$ of the symbol x_0 and consider any $x \in N_{x_0} \cap X - \{x_0\}$. Obviously $x \in X$. Moreover $x \in N_{x_0} =]\delta,+\infty[$, hence $x > \delta$. By (5.1.51), $f(x) > k = |h| + 1 > |h| \geq h$. Hence $f(x) \in]h,+\infty[= N_l$. ⋄

Theorem 5.1.7 *Let f be a real function of domain $X \subseteq \mathbb{R}$, $l = +\infty$. Let $x_0 = -\infty$ be an accumulation point of X. The following statements are equivalent:*

(5.1.52) $$\forall N_l \quad \exists N_{x_0} \;:\; \left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) \in N_l)$$

(5.1.53) $\forall k \in]0,+\infty[\quad \exists \delta \in]0,+\infty[\ : \ \forall x \in X$
$$(x < -\delta) \Rightarrow (f(x) > k)\ .$$

Proof. The demonstration is very similar to that of theorem 5.1.6. ◊

Theorem 5.1.8 *Let f be a real function of domain $X \subseteq \mathbb{R}$, $l = -\infty$. Let $x_0 = +\infty$ be an accumulation point of X. The following statements are equivalent:*

(5.1.54) $\forall N_l \quad \exists N_{x_0} \ : \ \left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) \in N_l)$

(5.1.55) $\forall k \in]0,+\infty[\quad \exists \delta \in]0,+\infty[\ : \ \forall x \in X$
$$(x > \delta) \Rightarrow (f(x) < -k)\ .$$

Proof. (5.1.54) $\Rightarrow$ (5.1.55). Suppose the (5.1.54) true. To obtain the (5.1.55), we choose any $k \in]0,+\infty[$. Then, $N_l =]-\infty,-k[$ is a neighborhood of the symbol l. By (5.1.54), it exists a neighborhood $N_{x_0}$of the symbol x_0 such that

(5.1.56) $$\left(x \in N_{x_0} \cap X - \{x_0\}\right) \Rightarrow (f(x) < -k).$$

Since N_{x_0} is a neighborhood of the symbol $x_0 = +\infty$, there exist $h \in \mathbb{R}$ such that $N_{x_0} =]h,+\infty[$. So, $\delta = |h| + 1 \in]0,+\infty[$. Choose $x \in X$ such that $x > \delta$. Hence $x > h$, hence $x \in N_{x_0}$. Since $x \notin \{+\infty\}$, it results $x \in N_{x_0} \cap X - \{x_0\}$. By (5.1.56), we have $f(x) < -k$.
(5.1.55) $\Rightarrow$ (5.1.54). Suppose the (5.1.55) true. To obtain the (5.1.54), we choose any neighborhood N_l of $l = -\infty$. Then, there exist $h \in \mathbb{R}$ such that $N_l =]-\infty,h[$. So, $k = |h| + 1 \geq 1 > 0$. Hence $k \in]0,+\infty[$. By (5.1.55), there exists $\delta \in]0,+\infty[$ such that $\forall x \in X$