

Aldo Maceri

Fluid Dynamics



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Aldo Maceri - *Fluid Dynamics*

PREFACE

The problem we are examining here is that of building a mathematical model that simulates the behavior of a *fluid*, which can be a gas or a liquid. In this analysis, *Thermodynamics* joins *Mechanics* with a greater role than that usually performed in the case of *solids*. We immediately point out that, even in the presence of thermal effects, the analysis of the main problems of engineering interest can be conducted by simulating the body with a continuous medium. However, we specify that in some engineering applications of *Fluid Dynamics* we encounter fluids for which the continuum model is not admissible (as in the case of *rarefied gases*).

In fluid dynamics, as a rule, non-negligible *mechanical energy dissipation* phenomena occur, with consequent production of *entropy*. If entropy is produced during a transformation, this cannot be canceled and it follows that the transformation considered is *irreversible*. It is therefore obligatory, in order to construct an adequate model of the problem, to resort to the *Thermodynamics of irreversible processes*. However, let us premise some references to *classical Thermodynamics*, based on a theory formulated with postulates.

This book is an introduction to *Fluid Dynamics*. It deduces its fundamental results favoring (when possible) considerations of a *physical nature*.

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Aldo Maceri

CONTENTS

Preface

<i>Chapter 1</i>	<i>Continuum Mechanics</i>	1
	1.1 Introduction	1
	1.2 Recalls of <i>classical Thermodynamics</i>	1
	1.3 The balance equations.....	14
	1.4 <i>Thermodynamics</i> of irreversible processes.....	18
<i>Chapter 2</i>	<i>Fluid Mechanics</i>	24
	2.1 The equations of <i>Fluid Dynamics</i>	24
	2.2 The characteristic numbers.....	32
	2.3 The nozzle.....	57
	2.4 Dissipative flows.....	82
	2.5 The laminar regime.....	94
	2.6 The turbulent regime.....	101
	<i>Bibliography</i>	109

Chapter 1

Mechanics of the continuum

1.1 Introduction.

The problem we will now consider is that of building a mathematical model that simulates the behavior of a fluid, which can be a gas or a liquid. In this analysis, *Thermodynamics* joins *Mechanics* with a greater role than that usually performed in the case of solids. We immediately point out that, even in the presence of thermal effects, the analysis of the main problems of engineering interest can be conducted by simulating the body with a continuous medium. Strictly speaking however, as is well known in the *Structure of matter*, the volume occupied by the atoms present in a body is only a very small part of the volume of the body. For example, in a solid-state piece of steel, the ratio is on the order of 10^{-14} .

It may happen that non-negligible mechanical energy dissipation phenomena occur in the problem under analysis. In this case, in order to construct an adequate mathematical model of the problem, it is necessary to resort to the *Thermodynamics of irreversible processes*.

1.2 Review of classical Thermodynamics. We call *system* a portion of space occupied by matter, *environment* the rest of the universe. We denote with $V(t)$ the portion of the three-dimensional space occupied by the system

at the instant $t \in]t_i, t_f[$ and with $\Sigma(t)$ its *boundary* (i.e. the *surface* of the system). The symbol t_i [resp. t_f] denotes the *initial instant* [resp. *final instant*] of the time interval in which we analyze the phenomenon.

We formulate a theory of *classical Thermodynamics* via postulates. We assume as *first postulate*

[1.1] *A system can exchange mass and energy with its environment.* ■

In [1.1] *mass* must be understood in its various forms (i.e. in its various states of aggregation) and *energy* in its various forms (potential, kinetic, electromagnetic, etc.). The postulate [1.1] states that everything that the system exchanges with the environment is associated with an exchange of mass and/or an exchange of energy. For example, the exchange of *mechanical momentum* is associated with an exchange of mass; the exchange of *electromagnetic momentum* is associated with an exchange of energy.

Clearly this first postulate of *Thermodynamics* is valid only outside the relativistic field. In fact, in the relativistic field it is not possible to exchange mass and energy independently.

Of course, admitting this first postulate of *Thermodynamics* implies that the system can exchange with the environment either energy but not mass or mass and energy. In fact, the exchanged mass brings with itself (even in a non-relativistic field) its energy.

A quantity is said to be *extensive* [resp. *intensive*] when its numerical value depends on [resp. does not depend on] the extension of the system.

Evidently if G is an *extensive quantity*, called g the *specific quantity* (also

called *density* of G), it results

$$G = \int_V g \, dV .$$

We call *punctual flow* (or *local flow*) at the instant $t \in]t_i, t_f[$ of an extensive quantity G through a surface $\Lambda(t)$ at a point $P(t)$ (of $\Lambda(t)$), and denote by the symbol φ_G , the quantity of G which crosses in the unit of time a unit of area of $\Lambda(t)$ to which $P(t)$ belongs. The surface $\Lambda(t)$ is supposed to be *regular*, so that each of its points admits a tangent plane (and a normal \mathbf{n}). Evidently

$$\varphi_G = g \, \mathbf{v} \times \mathbf{n} .$$

We call *flow* (or *global flow*) at the instant $t \in]t_i, t_f[$ of an *extensive quantity* G through a surface $\Lambda(t)$, and denote by the symbol $\Phi_G(t)$, the real number

$$\Phi_G(t) = \int_{\Lambda(t)} \varphi_G \, d\sigma = \int_{\Lambda(t)} g \, \mathbf{v} \times \mathbf{n} \, d\sigma .$$

We call *punctual production* (or *local production*) at the instant $t \in]t_i, t_f[$ of an *extensive quantity* G in a volume $V(t)$ at a point $P(t)$ (of $V(t)$), and we denote by symbol δ_G , the quantity of G which is created (or destroyed) in the unit of time in a unit of volume (of $V(t)$) to which $P(t)$ belongs.

We call *production* (or *global production*) at the instant $t \in]t_i, t_f[$ of an *extensive quantity* G in a volume $V(t)$, and denote by the symbol $\Delta_G(t)$,

the real number

$$\Delta_G(t) = \int_{V(t)} \delta_G dV .$$

A system is said to be in *thermodynamic equilibrium* if, at each of its points (*interior* or *boundary*), all local flows and all local productions are zero.

Let us now formulate the *second postulate* (which is usually called the *first law of Thermodynamics*).

[1.2] *There exists an extensive quantity U (which we call internal energy) and an extensive quantity S (which we call entropy) each of which is a function (in conditions of thermodynamic equilibrium) of a finite number of extensive quantities of the system, which we call state variables. ■*

The postulate [1.2] allows us to assume S [resp. U] as state variable. Therefore, called X_1, \dots, X_m ($m \in N$) the other state variables from which U [resp. S] depends, we have

$$U = U(S, X_1, \dots, X_m)$$

$$[\text{resp. } S = S(U, X_1, \dots, X_m)] .$$

We also say that the system has the $m + 1$ degrees of freedom S, X_1, \dots, X_m [resp. U, X_1, \dots, X_m].

We postulate now

[1.3] *The functions U and S are continuous, homogeneous of the first degree and it results*

$$\frac{\partial U}{\partial S} > 0, \quad \frac{\partial S}{\partial U} > 0.$$

Furthermore, in conditions of thermodynamic equilibrium, called T ^{1.1} the absolute temperature of the system, it results

$$(1.1) \quad \frac{\partial U}{\partial S} = T, \quad \frac{\partial S}{\partial U} = \frac{1}{T}. \quad \blacksquare$$

We explicitly note that, for the postulate [1.3], we have, $\forall \lambda \in \mathfrak{R}$

$$\begin{aligned} U(\lambda S, \lambda X_1, \dots, \lambda X_m) &= \lambda U(S, X_1, \dots, X_m) \\ S(\lambda U, \lambda X_1, \dots, \lambda X_m) &= \lambda S(U, X_1, \dots, X_m). \end{aligned}$$

Let us now formulate the *fourth postulate* (which is usually called the *second law of Thermodynamics*)

[1.4] *Internal energy U can be produced or destroyed. Entropy S can be produced but not destroyed^{1.2}. ■*

We now postulate (*third law of Thermodynamics* or *Nernst's*

^{1.1} Also called *thermodynamic*. The *absolute temperature scale* was established by *Lord Kelvin*, who assigned $T=273,16 \text{ }^\circ\text{K}$ to water and ice in conditions of thermodynamic equilibrium (at a pressure of 1 atm).

^{1.2} The second law of Thermodynamics also admits other equivalent formulations. Note that it is not possible to attach physical meaning to entropy. However, its importance is such that it is necessary to get used to considering it as an attribute of the system, just like mass or volume.

principle):

[1.5] *When T tends to zero S tends to zero.* ■

We now denote with E the *total energy* of the system (in all its forms) and we postulate that (*principle of conservation of energy*)

[1.6] *If a system is in thermodynamic equilibrium it results $\Delta_E = 0$.* ■

We call *transformation* of the system any evolution of the system that takes it from a state of thermodynamic equilibrium to another state of thermodynamic equilibrium. If the final state coincides with the initial one, the transformation is called *closed* or rather *cycle*. If the evolution of the system does not lead to a state of thermodynamic equilibrium or does not start from a state of thermodynamic equilibrium, it is preferable to speak of a *process* rather than a *transformation*.

A transformation is said to be *reversible* if it is possible to bring the system and the environment back to their initial conditions.

Generally, a cycle brings the system back to the initial conditions but not the environment. Therefore, it is reversible if it consists of two reversible transformations.

If entropy is produced during a transformation, this cannot be canceled and it follows that the transformation considered is *irreversible*.

A transformation (or process) is said to be *adiabatic* if (called Q the

flow of energy in the form of heat) is $Q = 0$.

A transformation (or process) is said to be *isentropic* if

$$\frac{dS}{dt} = 0 .$$

We consider a system in thermodynamic equilibrium. We know that it is completely characterized by $m \in \mathbb{N}$ state variables (in the sense that if the value of these m parameters is known, it is possible to determine the value of any thermodynamic quantity of the system). Therefore, the thermodynamic behavior of the system is fully characterized if m independent equations are known in the state parameters: the fundamental relation and $m - 1$ independent equations of state.

For a gas, consisting of a single chemical phase and a single physical phase, in conditions of thermodynamic equilibrium, the state parameters are the mass M , the volume V , the entropy S (so that $m = 3$). We know that the internal energy state function U exists, so that

$$U = U(S, V, M)$$

so

$$dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial V} dV + \frac{\partial U}{\partial M} dM .$$

We postulated that

$$\frac{\partial U}{\partial S} = T ;$$

moreover, since $\frac{\partial U}{\partial V} dV$ is an energy,

$$\frac{\partial U}{\partial V}$$

must be a *pressure* and we denote it with the symbol p . Similarly, $\frac{\partial U}{\partial M}$ is energy per unit mass and we call it the *electrochemical potential* μ .

Therefore, for the system under consideration the fundamental relation is (*Euler*)

$$(1.2) \quad dU = T dS + p dV + \mu dM$$

(note that, if the system does not exchange mass, the third addendum to the second side is zero).

If we add to the fundamental relation two equations of state

$$p = p(S, V, M)$$

$$\mu = \mu(S, V, M)$$

we fully identify the thermodynamic behavior of the system.

Normally the fundamental relation is obtained by hypothesizing a model (*i.e.* conjecturing an expression of it) and then going to verify what its field of validity is.

If a gas has r components (with r positive integer), in perfect analogy we obtain the fundamental equation (*Euler's*)

$$dS = \frac{1}{T} dU + \frac{p}{T} dV - \sum_{i=1}^r \frac{\mu_i}{T} dM_i$$

to which $r + 1$ equations of state must be added (since $m = r + 2$).

The equations of state express the extensive parameters as a function of the intensive parameters. An equation of state of a system composed of m grams of a gas of molecular weight M can be approximated by the law^{1.3}, setting $N = m/M$

$$(1.3) \quad pV = N R_0 T .$$

A substance which exactly obeys (1.3) is called *perfect gas* or *ideal gas*. All real gases have a behavior which is simulated by (1.3) only approximately.

The second equation of state of an ideal gas is

$$(1.4) \quad U = c_V T$$

^{1.3} In formula (1.2.3) R_0 denotes a *universal constant* for all gases, equal to $R_0 = 8316,6 \text{ N} \cdot \text{m}/^\circ\text{K} = 1,986 \text{ Cal}/^\circ\text{K}$. It is also used to place $R = R_0 / M$ in (1.2.3). Equation (1.2.3) summarizes *Boyle's* ($pV = \text{const}$), *Gay-Lussac's* ($V = V_0 \left(1 + \frac{T}{273,14}\right)$, $p = p_0 \left(1 + \frac{T}{273,14}\right)$) and *Avogadro's* (equal volumes of any gas, under equal conditions of temperature and pressure, contain the same number of molecules) laws.

where the constant c_V (which is called *specific heat at constant volume*) is equal to $\frac{3}{2} R$ for a monatomic gas, to $\frac{5}{2} R$ for a diatomic gas ^{1.4}.

Moreover, for an ideal gas it is easily obtained that, called c_p the *specific heat at constant pressure*, it results (*Mayer*)

$$c_p - c_V = R .$$

REMARK 1.1 We note that an *adiabatic transformation* of a perfect gas has the equation $p V^\gamma = \text{const}$ (being $\gamma = \frac{c_p}{c_V}$). In fact, we will see that an *adiabatic* is *isentropic*. Therefore from (1.2), (1.4) follows

$$c_V dT = p dV$$

and from here and from (1.3) we derive

$$\frac{dT}{T} + \frac{c_p - c_V}{c_V} \frac{dV}{V} = 0$$

from which

$$\frac{dT}{T} + (\gamma - 1) \frac{dV}{V} = 0$$

hence the thesis. ■

REMARK 1.2 The quotient

^{1.4} This result is obtained in a simple way using the *kinetic theory*.

$$c = \frac{q}{T_2 - T_1} ,$$

where q is the amount of heat required to bring the unit mass of the substance from T_1 to T_2 , is called the *specific heat c of a substance between the two (absolute) temperatures T_1 and T_2* .

Thus, for a body weighing P Kg, the quantity of heat necessary to heat it from T_1 to T_2 is

$$Q = cP(T_2 - T_1)$$

Note that the exact definition of c is

$$c = \lim_{T_2 \rightarrow T_1} \frac{q}{T_2 - T_1} = \frac{dq}{dT} . \blacksquare$$

In the case of gas, the *specific heat at constant pressure c_p* and the *specific heat at constant volume c_v* are important.

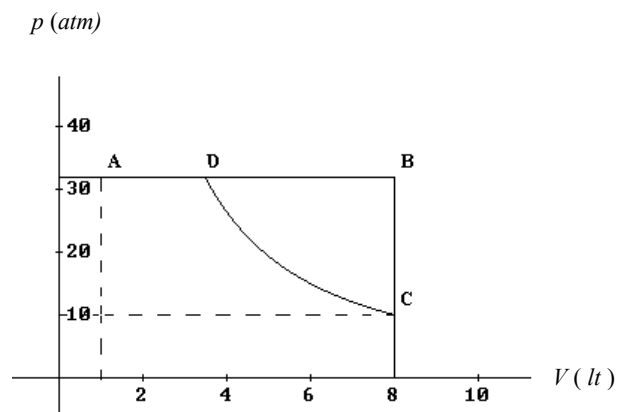


fig. 1.1

PROBLEM 1.1 A fluid (having $\gamma = 2,4$) evolves according to the transformation $A \rightarrow B \rightarrow C \rightarrow D$ of fig. 1.1, where the transformation $C \rightarrow D$ is adiabatic. Determine the work done and the heat exchanged.

Solution. The transformation $C \rightarrow D$ has the equation $pV^K = \text{const}$ (with $K = \gamma - 1 = 1,4$).

As for the work, we have (since in $B \rightarrow C$ the volume is constant)

$$\begin{aligned}
 L &= L_{AB} + L_{BC} + L_{CD} = \int_A^B p \, dV + \int_B^C p \, dV + \int_C^D p \, dV \\
 &= p \int_A^B dV + 0 + \int_C^D p_D V_D^K V^{-K} \, dV = p(V_B - V_A) + p_D V_D^K \int_C^D V^{-K} \, dV \\
 &= 32 \cdot 7 \, \text{atm} \cdot \text{lt} + p_D V_D^K \left[\frac{V^{-K+1}}{-K+1} \right]_C^D \\
 &= 224 \, \text{atm} \cdot \text{lt} + \frac{p_D V_D^K}{1-K} (V_D^{1-K} - V_C^{1-K}) \\
 &= 224 + \frac{p_D V_D^K V_D^{1-K} - p_C V_C^K V_C^{1-K}}{1-K} = 224 + \frac{p_D V_D - p_C V_C}{1-K} \, \text{atm} \cdot \text{lt} .
 \end{aligned}$$

From here (observing that from the $p_C V_C^{1,4} = p_D V_D^{1,4}$ follows $V_D = 2,7 \, \text{lt}$) we get

$$L = \frac{32 \cdot 2,7^{1,4} - 10 \cdot 8^{1,4}}{-0,4} = 224 - 176 = 48 \, \text{atm} \cdot \text{lt} .$$

As for the heat flux, let us first observe that for a real gas ^{1.5}

$$c_p = AR \frac{\gamma}{\gamma-1} , \quad c_V = AR \frac{1}{\gamma-1} .$$

In the case of air, resulting

$$A = \frac{1}{427} \frac{\text{cal}}{\text{Kgm}} , \quad R = 29,27 \frac{\text{Kg m}}{\text{Kg } ^\circ\text{C}} ,$$

we have

$$c_p = 0,239 \frac{\text{cal}}{\text{Kg } ^\circ\text{C}} , \quad c_V = 0,171 \frac{\text{cal}}{\text{Kg } ^\circ\text{C}} .$$

Furthermore, using the equation of state (1.3), we obtain

$$T_A = 11 \text{ } ^\circ\text{C} , \quad T_B = 87,5 \text{ } ^\circ\text{C} , \quad T_C = 27,5 \text{ } ^\circ\text{C} .$$

Thereby

$$\begin{aligned} Q &= Q_{AB} + Q_{BC} + Q_{CD} = Q_{AB} + Q_{BC} + 0 = c_p(T_B - T_A) + c_V(T_C - T_B) \\ &= 18,3 \frac{\text{cal}}{\text{Kg}} - 10,3 \frac{\text{cal}}{\text{Kg}} = 8 \frac{\text{cal}}{\text{Kg}} . \blacksquare \end{aligned}$$

PROBLEM 1.2 *Two systems are isolated from the external environment (by impermeable, anergotic and adiabatic walls) and are separated between them by an impermeable, anergotic, diabatic wall. System 1 consists of two*

^{1.5} For an ideal gas $A = 1$.

moles of He (which is monatomic), system 2 of three moles of O_2 . The total energy of the system is 6000 Kcal. Calculate U_1 and U_2 at equilibrium.

Solution. At equilibrium $T_1 = T_2$. Since

$$U_1 = \frac{3}{2} N_1 R_0 T_1 = \frac{3}{2} \cdot 2 \cdot 1,98 \cdot T_1 \text{ Kcal}$$

$$U_2 = \frac{3}{2} N_2 R_0 T_2 = \frac{5}{2} \cdot 2 \cdot 1,98 \cdot T_2 \text{ Kcal}$$

it must be at equilibrium

$$U_1 + U_2 = 6000 \text{ Kcal}$$

$$\frac{U_1}{3 \cdot 1,98} = \frac{U_2}{7,5 \cdot 1,98}$$

and then

$$U_1 = 1720 \text{ Kcal} , \quad U_2 = 4280 \text{ Kcal} . \quad \blacksquare$$

1.2 The balance equations. Let us consider a quantity G of the system (for example mass or energy), function of the time variable $t \in]t_i, t_f[$ ^{1.6}. We denote by g the quantity of G contained in the unit volume of the system. Therefore g is a function of the spatial variables x, y, z and of the time variable t and results

^{1.6} $t_i \in [0, +\infty[$ is the initial instant; t_f is the final instant (i.e. a real number greater than t_i or the symbol $+\infty$).

$$\forall t \in]t_i, t_f[\quad G(t) = \int_{V(t)} g(x, y, z, t) \, dx \, dy \, dz .$$

Evidently G can vary in the time interval $]t, t + dt[$ only if we introduce another quantity of G from the outside or if it is created or destroyed inside the system itself. So the change of G in the time interval $]t, t + dt[$ is equal to the flow of G (through $\Sigma(t)$) in the time interval $]t, t + dt[$ plus the production (which if negative is a destruction) of G in $V(t)$ in the time interval $]t, t + dt[$. Therefore in $]t, t + dt[$

$$(1.5) \quad \frac{dG}{dt} = \Phi_G + \Delta_G .$$

The (1.5) is called the *balance equation (of G)*.

The balance equation is also valid for thermodynamic quantities. However, when it is applied to thermodynamic quantities it is necessary to assume that the system is in thermodynamic equilibrium.

A quantity G such that $\Delta_G = 0$ is said to be *conservative*. If in (1.5) G is a conservative quantity, (1.5) is also called the *conservation equation* or *conservation principle*. Evidently in an *isolated system*^{1.7} every conservative quantity is constant.

^{1.7} A system is said to be isolated if $\forall t \in]t_i, t_f[$ the flow across the boundary $\Sigma(t)$ (of the volume $V(t)$ occupied by the system at time t) of any quantity is zero.

We apply the balance equation to the total energy E of the system, which by postulate [1.6] is a conservative quantity. We have

$$(1.6) \quad \frac{dE}{dt} = \Phi_E .$$

We postulated that the system can exchange with the environment either *energy but not mass* or *energy and mass*. We then split Φ_E into the sum of Φ_{EE} (energy flux that is not associated with mass exchange) and Φ_{EM} (energy flux associated with mass exchange).

It is convenient (and it will be clear later why) to consider the energy flow Φ_{EE} sum of an aliquot Q (which we call *heat flow* per unit of time or *thermal power* or simply *heat*) and of an aliquot L (which we call *work flow* per unit of time or *mechanical power* or simply *work* (per unit of time)).

Therefore, if Σ does not allow mass exchange between system and environment, it results

$$(1.7) \quad \frac{dE}{dt} = Q + L .$$

In (1.7) (which is one of the forms of the *principle of conservation of energy*) it should be highlighted that Q and L are two forms of *energy flows*, *i.e.* they are both *energies in transit*.

The variation of E in a time interval $[t_1, t_2]$ is obtained by integrating the two sides of (1.7) with respect to time (from t_1 to t_2).

We now carry out the balance of a quantity that is not conserved. A typical quantity of thermodynamics that is not conserved is *entropy*. The

internal energy U is also not conserved. But we know something more about entropy: not only is it not conserved but it can't even be destroyed.

We immediately observe that since the quantity we want to balance is a thermodynamic quantity, in order to be able to define it, the system must be in thermodynamic equilibrium. So, from (1.5) we get ^{1.8} that in $]t_i, t_f[$

$$(1.8) \quad \frac{dS}{dt} - \Phi_S = \Delta_S .$$

By the postulate [1.4] $\Delta_S \geq 0$. So, the (1.8) gives

$$(1.9) \quad \frac{dS}{dt} - \Phi_S \geq 0 .$$

We postulated that the system can exchange with the environment either *energy but not mass* or *mass and energy*. Therefore, as for the total energy, we split Φ_S into the sum of an entropy flow Φ_{SM} associated with mass exchange and an entropy flow Φ_{SQ} associated with energy exchange but not mass exchange. Of course, if Σ does not allow mass exchange then $\Phi_{SM} = 0$.

We now give a fundamental definition of *Thermodynamics* which defines *heat* and *work a posteriori*. Φ_{SQ} is associated with the flow of energy

^{1.8} In this simplified formulation there is an inconsistency, because in conditions of thermodynamic equilibrium the flows and the productions are zero. However, in the exact setting, which preliminarily gives all the concepts of *equilibrium Thermodynamics*, this inconsistency does not exist.

in the form of heat. *Thus, that particular form of energy flow which we have called work does not bring, associated with itself, flow of entropy.* In conclusion, entropy can flow (through Σ) together with mass or together with energy or together with both, but it cannot flow alone (through Σ). So, if the system is isolated it results $\Phi_S = 0$.

We denote by φ_{SQ} the *local flow of entropy* (through Σ) *not associated with mass flow*, so that

$$\Phi_{SQ} = \int_{\Sigma} \varphi_{SQ} d\sigma .$$

As we can rigorously demonstrate, if T is the *absolute* (or *thermodynamic*) *temperature* of the system, it results

$$T \varphi_{SQ} = q .$$

In particular, if T is constant on Σ , we have

$$\Phi_{SQ} = \frac{Q}{T} .$$

1.3 Thermodynamics of irreversible processes. The *Thermodynamics of reversible processes* (which is also called *classical Thermodynamics*) correlates a state of thermodynamic equilibrium with another state of thermodynamic equilibrium, however reached. It states that if the energy is varied this is due to flows of heat and work; that if the entropy has varied

this is due to flows or productions of entropy. But it does not give information on what happens during the transformation if it does not go, instant by instant, through states of thermodynamic equilibrium (which are states in which flows and productions are identically zero).

In order to give meaning to flows and productions, and to derive operational engineering expressions from them, we build a new building (which includes the previous one of *classical Thermodynamics*), still on a postulatoric basis. First of all, let us formulate a new postulate that allows us to define internal energy, entropy and other thermodynamic quantities even in states of the system that are not in thermodynamic equilibrium.

[1.7] *At any point P of a system, called I a suitably small neighborhood of P , the subsystem identified by I is in a state of thermodynamic equilibrium.*

■

The main limitation of this postulate is that it implies that the history of the system, *i.e.* the processes previously undergone, do not influence the state of the system. In fact, it says that the internal energy in a point depends only on the values that the thermodynamic parameters have in that point. In nature, however, there is a vast phenomenology to which this postulate cannot be applied. For example in *hysteresis processes* and *heat treatments* the material properties depend on the previous history.

The postulate [1.7] allows to define the quantities of *classical Thermodynamics* even in states that are not in thermodynamic equilibrium. We now need tools that allow us to evaluate the flows and productions of

these quantities. We therefore postulate now (*Curie principle*)

[1.8] *If the system is isotropic each flow depends on all and only the forces generalized by the same tensor order ^{1.9} of the flow. ■*

Considering that in the following we will always deal with isotropic systems, and that we call *generalized force* ^{1.10} any cause that can cause flows, the postulate [1.8] tells us that a flow which is a scalar [resp. vector] [resp. tensor] depends on all generalized forces which are scalars [resp. vectors] [resp. tensors] and only from those.

For example, in combustion, chemical affinity is a generalized force of the scalar type. It pushes the reaction one way or the other. However, being a scalar, it cannot influence (if the medium is isotropic) neither the energy flows nor the mass flow (which are vector quantities).

Normally a generalized force causes one flow directly and other flows indirectly. For example, the generalized force $\frac{\partial}{\partial r} \frac{1}{T}$ is the direct cause of the flow of energy in the form of heat and is an indirect cause of the flow of mass.

What is the functional dependence of the flows on the causes remains to be clarified. We recall that (under suitable hypotheses) a real function of

^{1.9} A quantity (associated to a point) is said to be *scalar* (or *tensor of order zero*) if it is identified by a real number; *vectorial* (or *tensor of order 1*) if it is identified by an ordered triad of real numbers; *tensor* (or *tensor of order 2*, or simply *tensor*) if it is identified by an ordered sextuple of real numbers (that is, by two vectors). It should be noted that a direction is associated with a vector; two directions are associated with a tensor (or, if you prefer, a plane and a direction).

^{1.10} This is because the *work flow* is caused by a cause that we call a *force*.

a real variable can be developed in a *McLaurin* series

$$(1.10) \quad f(c) = f(0) + f'(0) c + f''(0) \frac{c^2}{2} + \dots .$$

We express for the effect (*i.e.* for the flow) this functional dependence on the cause (*i.e.* on the generalized force). We choose the cause so that when the cause is zero the effect is zero. Therefore $f(0) = 0$. We limit the study to the linear thermodynamics of irreversible processes (so that in (1.10) we neglect the terms of order greater than 1).

We have come to the conclusion that the effect is proportional to the cause and, most importantly, the coefficient (of proportionality) is independent of the cause. Therefore $f'(0)$ is a purely thermodynamic quantity, *i.e.* it is a function of state. For example, in *Fourier's* law of heat transfer

$$(1.11) \quad Q = k \frac{\partial T}{\partial r}$$

the constant k is a state function of the medium (of course this law is valid only if there is the possibility of mass flows).

Thus, in an isotropic system the fluxes depend on all the generalized forces of the same tensor order (direct and indirect). If the phenomenology is linear then the flows are proportional to all the causes that can cause them. Furthermore, the proportionality coefficients are state functions, *i.e.* they depend only on the state of the system.

Let us consider a system consisting of a mixture of two gases. There is energy flow in the form of heat Q and mass flow Φ_{M_1} of the first gas. The relative direct generalized forces are $\frac{\partial}{\partial r} \left(\frac{1}{T} \right)$ and $\frac{\partial}{\partial r} \left(\frac{\mu}{T} \right)$. According to *Curie's* principle, since these directed generalized forces are vectors and both fluxes are vectors, each flux must depend on both generalized forces. Then, in the field of linear thermodynamics we will write

$$Q = L_{qq} \frac{\partial}{\partial r} \left(\frac{1}{T} \right) + L_{qm} \frac{\partial}{\partial r} \left(\frac{\mu}{T} \right)$$

$$Q = L_{mq} \frac{\partial}{\partial r} \left(\frac{1}{T} \right) + L_{mm} \frac{\partial}{\partial r} \left(\frac{\mu}{T} \right) .$$

The direct proportionality coefficients L_{qq} , L_{mm} and the cross ones L_{qm} , L_{mq} are functions only of the state of the system.

The following *Onsager* theorem holds. It has general validity, even for systems with n degrees of freedom.

[1.9] *In the absence of magnetic fields the crossed coefficients are equal:*

$$L_{qm} = L_{mq} . \blacksquare$$

These conclusions are of the utmost practical importance. For example, it is thanks to them that it is possible to build refrigerators based on the thermoelectric effect. In them the exchange of energy in the form of heat is obtained by means of an electric potential difference.

We can prove that

$$(1.12) \quad \delta_S = \sum_{i=1}^n \Phi_i \cdot F_i .$$

In (1.12) δ_S is the local entropy production; Φ_1, \dots, Φ_n are all the fluxes involved; F_1, \dots, F_n are all the generalized forces involved. In (1.12) the products are scalar if flows and forces are vectors; scalar double products if flows and forces are tensors. We observe that in (1.12) the flow of energy in the form of work does not appear because it is not accompanied by the production of entropy.

We postulated that $\delta_S \geq 0$. This results in conditions on the direct and cross coefficients L_{ij} . For example, in the already examined case in which they are $L_{qq}, L_{mm}, L_{qm}, L_{mq}$ it results:

$$L_{qq} \geq 0$$

$$L_{mm} \geq 0$$

$$L_{qq} L_{mm} - L_{mq}^2 \geq 0 .$$

Capitolo 2

Meccanica dei fluidi

2.1 The equations of *Fluid Dynamics*. We begin the study of *Fluid Dynamics* with the following premises. Let \mathbf{u} and \mathbf{v} be two vectors of \mathfrak{R}^3 with components (u_x, u_y, u_z) , (v_x, v_y, v_z) . It is called the *tensor product* of \mathbf{u} by \mathbf{v} (and is denoted by $\mathbf{u} \cdot \mathbf{v}$) the tensor of \mathfrak{R}^3

$$\begin{bmatrix} u_x v_x & u_x v_y & u_x v_z \\ u_y v_x & u_y v_y & u_y v_z \\ u_z v_x & u_z v_y & u_z v_z \end{bmatrix}.$$

Let us consider a *tensor* $\boldsymbol{\tau}$ of \mathfrak{R}^3

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

and a *vector* \mathbf{n} of \mathfrak{R}^3

$$\mathbf{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}.$$

We call the *scalar product* of $\boldsymbol{\tau}$ by \mathbf{n} (and denote by the symbol $\boldsymbol{\tau} \times \mathbf{n}$) the vector of \mathfrak{R}^3

$$\mathbf{p} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \tau_{xx}n_x & \tau_{xy}n_y & \tau_{xz}n_z \\ \tau_{yx}n_x & \tau_{yy}n_y & \tau_{yz}n_z \\ \tau_{zx}n_x & \tau_{zy}n_y & \tau_{zz}n_z \end{bmatrix} .$$

Calling $\mathbf{x}, \mathbf{y}, \mathbf{z}$ the *unit vectors* of the coordinate axes (so that $\mathbf{p} = p_x\mathbf{x} + p_y\mathbf{y} + p_z\mathbf{z}$), we call the *flux of the tensor $\boldsymbol{\tau}$ through Σ* the vector

$$\begin{aligned} \int_{\Sigma} \boldsymbol{\tau} \times \mathbf{n} \, d\sigma &= \int_{\Sigma} p \, d\sigma = \\ &= \left(\int_{\Sigma} p_x \, dx \right) \mathbf{x} + \left(\int_{\Sigma} p_y \, dy \right) \mathbf{y} + \left(\int_{\Sigma} p_z \, dz \right) \mathbf{z} . \end{aligned}$$

After that, we highlight that any fluid dynamics problem has two types of unknowns: the state unknowns (which characterize the thermodynamic state of the system) and the kinetic unknown (which is the velocity vector). Obviously, these unknowns are functions of point and time.

It goes without saying that it is necessary to formulate as many scalar equations as there are the thermodynamic unknowns of state plus a vector equation in which the velocity vector intervenes. To formulate these equations, we impose the fact that the evolution of the system takes place by satisfying the conservation or balance equations.

If the system consists of only one phase, we impose conservation of mass. If instead it is a mixture of different masses, we also impose the

balance of the single masses (or the conservation of the single masses, if the possibility of chemical reactions is excluded). Another scalar equation is that of the conservation of total energy. A vector equation (which breaks down into as many scalar equations as there are components of the vector) is the momentum balance. The production of momentum is due to mass forces (such as the force of gravity). In the absence of such forces, the conservation of momentum will prevail.

Let us therefore explain the balance of the total mass M . Calling ρ the mass density, \mathbf{v} the velocity and \mathbf{n} the unit vector of a straight-line n , we have

$$G = M, \quad g = \rho, \quad \varphi_M = \rho \mathbf{v} \times \mathbf{n}, \quad \delta_M = 0.$$

Since the local production δ_M is zero, the balance becomes a conservation and is written

$$(2.1) \quad \frac{\partial}{\partial t} \int_V \rho \, dV + \int_{\Sigma} \rho \mathbf{v} \times \mathbf{n} \, d\sigma = 0.$$

Reasoning in the same way, applying the mass balance to an elementary parallelepiped (with faces parallel to the coordinate planes) and taking into account the divergence theorem, we obtain the differential equation in $V \times [t_i, +\infty[$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0.$$