SUBJECT MATTER

This chapter 1 opens the study of the *Mathematical Analysis*. First of all, they are clarified some basic principles of the mathematical logic. Still preliminarily, the significance of the main symbols used in the discipline (as $\in$, $\notin$, $\subseteq$, $\subset$, $\cup$, $\cap$, $\exists$, $:$, $\Rightarrow$, $\Leftrightarrow$, $\emptyset$) is clarified.

The study begins by defining the *set* and the *operations* on sets. After that, there are given the definitions of *cover, partition, Cartesian product.*

After such basic definitions, there are defined the *relations* on sets. Particular attention is devoted to *equivalence relation* (and to *quotient set*) and to *partial* or *total order relation*. For ordered sets, there are given the definitions of *upper bound, lower bound, max, min, least-upper-bound property, inf.*

After that, it is defined the *function*, also called *single-valued relation* or *mapping* or *transformation* or *operation* or *correspondence* or *application*. Still, there are defined the *reversible function*, the *inverse function*, the *one to one* (or *biunique*) *correspondence*, the *restriction function*, the *extension function*, the *composite function*, the *characteristic function*. In particular, the ono to one function allows to define the *equivalent sets*, the *finite set*, the *infinite set*, the *denumerable* (or *enumerable*) *set*, the *countable set*.

The basic study of sets is completed by giving some useful definitions and properties of the *algebra of sets*. Precisely, after the definition of *field*, there are detailed the *field axioms* and all the statements which they imply. Afterwards, there are given the definition of *ordered field* and all the statements which it implies.

The study of sets is completed by the construction of the most important set, that is the *real number set* $\mathbb{R}$. Such construction can be performed only using the set theory. However, the real number set is often constructed beginning by the *natural number set*

$\mathbb{N} = \{1,2,3, \dots\}$. In such way, the definition and the properties of $\mathbb{N}$ are assumed axiomatically, as primitive notions. After that, the natural number set is provided with the *laws of addition and multiplication*, and with the *order relation*.

Using $\mathbb{N}$ as foundation, we can build a new set Q, called *rational number set*, which can be considered an extension of $\mathbb{N}$. Preliminary, it is appropriate to build the set Q^+, called *positive rational number set.* For this purpose, we provide $\mathbb{N} \times \mathbb{N}$ with the equivalence relation

$$R = \{((n_1, n_2), (m_1, m_2)) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}): n_1 m_2 = n_2 m_1\}.$$

In doing this, to indicate the generic ordered pair of natural numbers (n, m), usually you prefer to use the $\frac{n}{m}$ notation. The equivalence relation R divide $\mathbb{N} \times \mathbb{N}$ into equivalence classes, in such a way that $\frac{n_1}{n_2} \equiv \frac{m_1}{m_2}$ if and only if $\frac{n_1}{n_2}$ and $\frac{m_1}{m_2}$ belong to the same class. The family of all such equivalence classes is a family of nonvoid pairwise disjoint sets and its union is $\mathbb{N} \times \mathbb{N}$ and then is a partition of $\mathbb{N} \times \mathbb{N}$ (called the *quotient set of* $\mathbb{N} \times \mathbb{N}$). Well, Q^+ is defined as such *quotient set*. It is provided with the laws of addition and multiplication and with an order relation. From Q^+ we can build a new set Q (containing Q^+), which we call *rational number set*. Analogously, we can build a new set I (containing I^+), which we call *integer number set*. Precisely, we create a new number 0 called *zero*, distinct from the positive rational numbers. We also create numbers which are distinct from the rational positive numbers as well as distinct from zero, and which we call *negative rational numbers.* The set consisting of all negative rational numbers, is denoted by Q^-. We call *rational number set*, and denote by Q, the set consisting of all positive rational numbers, of 0, and of all negative rational numbers. We call *integer number set*, and denote I, the set consisting of all positive integer numbers, of 0, and of all negative integer numbers. Obviously $I \subset Q$. Q, provided with the laws of addition and multiplication and with an useful order relation, is an *ordered field*.

The set Q, although it is an orderly field, has serious insufficiencies. The most insufficiency of Q is the lack of *completeness*. Precisely, we prove that Q has not the *least-upper-bound property*, also called the *completeness property*. The *completeness* is the fundamental property that makes possible the *Mathematical Analysis*. For this reason, it was created from Q a new number set, called *real number set*, and denoted $\mathbb{R}$ (or $]-\infty, +\infty[$). One of the ways to create $\mathbb{R}$ is due to *Dedekind.* He called *cut*, or *real number*, any nonempty subset x of Q such that $(p \in x,\ q \in Q,\ q < p) \Rightarrow (q \in \alpha)$ and $(p \in x) \Rightarrow (\exists r \in \alpha$ such that $p < r)$. About the order relation on $\mathbb{R}$, we say that x is *less than* y (or that y is *greater than* x) and we write $x < y$ (or $y > x$) if x is a proper subset of y. After that, we prove that the ordered set $\mathbb{R}$ has the *least-upper-bound property.* Moreover, providing $\mathbb{R}$ with suitable operations of *addition, multiplication, subtraction, division*, we can demonstrate that $\mathbb{R}$ is a *complete ordered field.* To complete the study of $\mathbb{R}$, we demonstrate its *Archimedean property* and define the *rational real numbers*, the *irrational real numbers*, the *absolute value* of a real number (and detail its properties).

It is convenient to adjoin to $\mathbb{R}$ two new elements, *which are not real numbers*. These objects, called *infinity* and *minus infinity*, are respectively denoted $+\infty$ (or simply ∞) and $-\infty$. The set $[-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$ is called the *extended real number set*. We provide it with an order and with operations of addition and of multiplication.

As useful tools, there are defined

- the *open interval* $]a, b[$ having *left endpoint* a and *right endpoint b*
- the *closed interval* $[a, b]$ having *left endpoint* a and *right endpoint b*
- the *integral power* of a real number (with the detail of its properties)
- the *n-th root* of a real number
- the *factorial* of $n \in \mathbb{N} \cup \{0\}$

- the *binomial coefficient* $\binom{n}{k}$,

there are proven

- the Newton's binomial formula
- the density of Q in $\mathbb{R}$

and it is discussed the *representation* (in various *bases*) of the numbers.

Another important number set is the set consisting of all ordered pairs of real numbers, called *complex number Set.* This set, provided with suitable operations of addition and multiplication, is a *field*. However, it *cannot be ordered*. Then, it makes no sense to talk about inequalities between complex numbers.

Introducing the imaginary unit i, any *complex number* (a, b) can take the *algebraic form* $a + ib$, where a is called the *real part* and b is called the *imaginary part*. After that, for any complex number there are given the definitions of *complex conjugate* and of *absolute value* (or *modulus*).

There is also another form of complex numbers, very much employed in various sectors of *Engineering*. It is called *trigonometric form* of the complex number. Precisely, with each complex number $z = a + ib$ we associate a point with rectangular coordinates a and b in the complex plane. The x-axis along which a is reckoned is called *real axis* (or the *axis of reals*) and the y-axis along which b is reckoned is called *imaginary axis* (or the *axis of imaginaries*). Denoting by ϱ the length of the radius vector of the point z and by ϑ (for $z \neq 0$) the angle (expressed in radians) formed by the radius vector with the positive half-axis x we clearly have $a = \varrho \cos\vartheta$, $b = \varrho \sin\vartheta$, $\varrho = \sqrt{a^2 + b^2} \geq 0$. Hence, since $z = a + ib$, we have $z = \varrho(\cos\vartheta + i\ \sin\vartheta) = [\ \varrho, \vartheta]$. The trigonometric form allows to perform in a very simple way the operations of multiplication, division, calculation of integral power, calculation of nth roots. In particular, there are given all the nth complex roots of the positive real unit.